The six-pion amplitude

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Introduction

In low-energy region, we cannot study perturbatively the interactions of hadrons directly from QCD
$\hookrightarrow$ alternative approaches $\rightarrow$ Chiral perturbation theory (ChPT)
Weinberg, Phys.A 96, (1979), *Gasser and Leutwyler*, Ann.Ph.158 (1984)

Many observables are known in ChPT to a high loop order
$\hookrightarrow$ only recently it has become of interest to calculate the six-pion amplitude at low energies after it has been estimated using lattice QCD
Blanton et al., PRL 124 (2020), JHEP 10 (2021),
Fischer et al., EPJC 81 (2021), *Hansen et al.*, PRL 126 (2021),
Brett et al., PRD 104 (2021)

The six-pion amplitude at tree level was first done using current algebra methods
e.g. *Osborn*, Lett.N.Cim.2 (1969)
It has been redone with Lagrangian methods many times, not known to one-loop order
e.g. *Low et al.*, JHEP 11 (2019), *Bijnens et al.*, JHEP 11 (2019)

We have therefore calculated at NLO the six-pion amplitude
$\hookrightarrow$ within ChPT generalization to the O$(N+1)$/O(N) massive nonlinear sigma model
$\hookrightarrow$ two(-quark)-flavour ChPT equivalent to O(4)/O(3)

Massive $\mathrm{O}(N+1)/\mathrm{O}(N)$ nonlinear sigma model extended beyond the LO

$$\mathcal{L} = \frac{F^2}{2}\,\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi + F^2\chi^\mathsf{T}\Phi$$
$$+\,l_1\left(\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi\right)\left(\partial_\nu\Phi^\mathsf{T}\partial^\nu\Phi\right) + l_2\left(\partial_\mu\Phi^\mathsf{T}\partial_\nu\Phi\right)\left(\partial^\mu\Phi^\mathsf{T}\partial^\nu\Phi\right) + l_3\left(\chi^\mathsf{T}\Phi\right)^2 + l_4\partial_\mu\chi^\mathsf{T}\partial^\mu\Phi$$

Φ: real vector of $N+1$ components, $\Phi^\mathsf{T}\Phi = 1$
$\chi^\mathsf{T} = \left(M^2, \vec{0}\right)$

F, M: bare pion decay constant and mass

$\hookrightarrow$ calculate the four-pion and six-pion amplitudes at NLO

$$\Phi_1 = \left(\sqrt{1-\varphi},\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$ *Gasser and Leutwyler*, Ann.Ph.158 (1984)

$$\Phi_2 = \frac{1}{\sqrt{1+\varphi}}\left(1,\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$ simple variation

$$\Phi_3 = \left(1-\frac{1}{2}\varphi,\, \sqrt{1-\frac{1}{4}\varphi}\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$ ESB term only gives mass terms of ϕ_is

$$\Phi_4 = \left(\cos\sqrt{\varphi},\, \frac{1}{\sqrt{\varphi}}\sin\sqrt{\varphi}\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$ follows the general prescription from *Coleman, Wess and Zumino*, PR 177 (1969)

$$\Phi_5 = \frac{1}{1+\frac{1}{4}\varphi}\left(1-\frac{1}{4}\varphi,\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$ *Weinberg*, PR 166 (1968)

$\varphi \equiv \dfrac{\boldsymbol{\phi}^\mathsf{T}\boldsymbol{\phi}}{F^2}$, with $\boldsymbol{\phi}^\mathsf{T} = (\phi_1, \ldots, \phi_N)$ a real vector of N components (flavours)

$\hookrightarrow$ few examples of the whole class of parametrizations

$$\Phi = \left(\sqrt{1-\varphi\, f^2(\varphi)},\, f(\varphi)\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T},$$ with $f(x)$ any analytical function satisfying $f(0) = 1$

Four-pion amplitude

On-shell amplitude in general

p_i, $i = 1, \ldots, 4$ pion incoming four-momenta, $\sum p_i = 0$

f_i flavours

Invariance under rotation in the isospin space and crossing symmetry implies

$$A_{4\pi}(p_1, f_1, p_2, f_2, p_3, f_3, f_4) \\ = \delta_{f_1 f_2}\delta_{f_3 f_4} A(p_1, p_2, p_3) + \delta_{f_1 f_3}\delta_{f_2 f_4} A(p_3, p_1, p_2) + \delta_{f_2 f_3}\delta_{f_1 f_4} A(p_2, p_3, p_1)$$

Mandelstam variables

$s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2$, $s + t + u = 4M^2$

$\hookrightarrow$ subamplitude $A(p_1, p_2, p_3) = A(s, t, u)$

Four-pion amplitude
Leading order

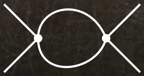
The leading-order $\mathcal{O}(p^2)$ amplitude stems from a single diagram

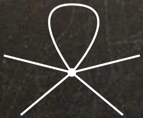
LO subamplitude (with LO relations $M \to M_\pi$ and $F \to F_\pi$)

$$A^{(2)}(s,t,u) = \frac{1}{F_\pi^2}\left(s - M_\pi^2\right)$$

Four-pion amplitude
Next-to-leading order

At NLO, one-loop diagrams (two topologies of 4 one-loop diagrams in total) and a counterterm


(a) $3\times$


(b) $1\times$


(c) $1\times$

$+$ NLO field renormalization, and mass and decay-constant redefinitions applied to the LO graph
$\hookrightarrow$ schematically $A^{(4)}_{4\pi} = \mathcal{M}_{\text{1-loop}} + \mathcal{M}_{\text{CT}} + 4(Z^{1/2}-1)\mathcal{M}^{(2)}_{\text{LO}} + \mathcal{M}^{(4)}_{\text{LO}}$

The Z factor is related to the pion self-energy Σ: $\qquad \dfrac{1}{Z} = 1 - \dfrac{\partial\Sigma(p^2)}{\partial p^2}\bigg|_{p^2=M_\pi^2}$

Standard relations $M_\pi^2 = M^2 - \overline{\Sigma}$, $F_\pi = F(1+\delta F)$ give the substitutions at the given order

$$M^2 \to M_\pi^2 + \overline{\Sigma}\,, \qquad \overline{\Sigma} = \frac{M_\pi^4}{F_\pi^2}\left[2l_3^{\mathrm{r}} + \frac{1}{2}(N-2)L\right] + \mathcal{O}\left(\frac{1}{F_\pi^4}\right)$$

$$\frac{1}{F^2} \to \frac{1}{F_\pi^2}(1+2\delta F)\,, \quad \delta F = \frac{M_\pi^2}{F_\pi^2}\left[l_4^{\mathrm{r}} - \frac{1}{2}(N-1)L\right] + \mathcal{O}\left(\frac{1}{F_\pi^4}\right)$$

Parametrization-independent and UV-finite result

$$\begin{aligned}
F_\pi^4 A^{(4)}(s,t,u) = {} & (t-u)^2\left(-\frac{5}{36}\,\kappa - \frac{1}{6}\,L + \frac{1}{2}\,l_2^{\mathrm{r}}\right) \\
& + M_\pi^2 s\left[\left(N-\frac{29}{9}\right)\kappa + \left(N-\frac{11}{3}\right)L - 8l_1^{\mathrm{r}} + 2l_4^{\mathrm{r}}\right] \\
& + s^2\left[\left(\frac{11}{12}-\frac{N}{2}\right)\kappa + \left(1-\frac{N}{2}\right)L + 2l_1^{\mathrm{r}} + \frac{1}{2}\,l_2^{\mathrm{r}}\right] \\
& + M_\pi^4\left[\left(\frac{20}{9}-\frac{N}{2}\right)\kappa + \left(\frac{8}{3}-\frac{N}{2}\right)L + 8\,l_1^{\mathrm{r}} + 2l_3^{\mathrm{r}} - 2l_4^{\mathrm{r}}\right] \\
& + \bar{J}(s)\left[\left(\frac{N}{2}-1\right)s^2 + (3-N)M_\pi^2 s + \left(\frac{N}{2}-2\right)M_\pi^4\right] \\
& + \left\{\frac{1}{6}\,\bar{J}(t)\left[2t^2 - 10M_\pi^2 t - 4M_\pi^2 s + st + 14M_\pi^4\right] + (t\leftrightarrow u)\right\}
\end{aligned}$$

Above we used

$$\kappa = \frac{1}{16\pi^2}\,, \quad L \equiv \kappa\log\frac{M_\pi^2}{\mu^2}\,, \quad \bar{J}(q^2) \equiv \kappa\left(2+\sigma\log\frac{\sigma-1}{\sigma+1}\right), \quad \sigma = \sqrt{1-\frac{4M_\pi^2}{q^2}}$$

Form as given in *Bijnens et al.*, PLB 374 (1996), NPB 508 (1997), generalized to $N \neq 3$

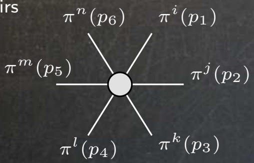
The expressions agree with the known results
↪ for $N=3$, equivalent (to a given order) to *Gasser and Leutwyler*, Ann.Ph.158 (1984)
↪ on the N dependence, e.g. *Dobado and Morales*, PRD 52 (1995),
Bijnens and Carloni, NPB 827 (2010), NPB 843 (2011)

Six-pion amplitude

4 pions $\to$ 3 channels/permutations/ways to distribute 4 pions in 2 pairs

6 pions $\to$ 10 ways in 2 groups of three (P_{10})

$\hookrightarrow$ 15 ways in 3 pairs (P_{15})



The full six-pion amplitude at $\mathcal{O}(p^4)$

$$A_{6\pi} = A_{6\pi}^{(4\pi)} + A_{6\pi}^{(6\pi)}$$

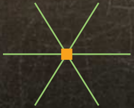
$A_{6\pi}^{(4\pi)}$ can be written in terms of the four-pion amplitude and $A_{6\pi}^{(6\pi)}$ is the remainder

(a) 1×

(b) 10×

Six-pion amplitude

Feynman diagrams of relevant topologies


(a) 1×


(b) 1×

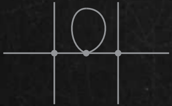

(c) 15×


(d) 20×


(e) 20×


(f) 60×


(g) 10×


(h) 10×


(i) 15×

Six-pion amplitude
The pole part

$$A_{6\pi}^{(4\pi)} \equiv \sum_{P_{10}, f_\circ} A_{4\pi}(p_i, f_i, p_j, f_j, p_k, f_k, f_\circ) \, \frac{(-1)}{p_{ijk}^2 - M_\pi^2} \, A_{4\pi}(p_l, f_l, p_m, f_m, p_n, f_n, f_\circ)$$

$\hookrightarrow$ residue at the pole unique, off-shell extrapolation away from $p_{ijk}^2 \equiv (p_i + p_j + p_k)^2 = M_\pi^2$ not

$A_{4\pi}(p_i, f_i, p_j, f_j, p_k, f_k, f_\circ)$ is the four-pion amplitude with one leg off-shell

$$A_{4\pi}(p_i, f_i, p_j, f_j, p_k, f_k, f_\circ)$$
$$= \delta_{f_i f_j} \delta_{f_k f_\circ} A(p_i, p_j, p_k) + \delta_{f_i f_k} \delta_{f_j f_\circ} A(p_k, p_i, p_j) + \delta_{f_j f_k} \delta_{f_i f_\circ} A(p_j, p_k, p_i)$$

The (four-pion) subamplitude $A(p_i, p_j, p_k) = A(s, t, u)$ is defined as usual
$\hookrightarrow$ $s = (p_i + p_j)^2$, $t = (p_i + p_k)^2$ and $u = (p_j + p_k)^2$, although now $s + t + u = 3M_\pi^2 + p_{ijk}^2$

We have chosen a particular form for the off-shell four-pion subamplitude $A(s, t, u)$
$\hookrightarrow$ other off-shell extrapolations are possible and will lead to a different $A_{6\pi}^{(6\pi)}$
$\hookrightarrow$ $A_{6\pi}^{(4\pi)}$ independent of the parametrization used, consequently also $A_{6\pi}^{(6\pi)}$

$$A^{(6\pi)}_{6\pi} \equiv \sum_{P_{15}} \delta_{f_i f_j} \delta_{f_k f_l} \delta_{f_m f_n} A(p_i, p_j, p_k, p_l, p_m, p_n)$$

The (six-pion) subamplitude $A(p_1, p_2, p_3, p_4, p_5, p_6)$

$\hookrightarrow$ no poles, only cuts (however, the imaginary part of the triangle integrals can contain poles)

$\hookrightarrow$ function of three pairs of momenta

- $\hookrightarrow$ fully symmetric under the interchange of any of the pairs
- $\hookrightarrow$ symmetric for the interchange within a pair

(a) 1× (b) 10×

The full six-pion amplitude at $\mathcal{O}(p^4)$

$$A_{6\pi} = A_{6\pi}^{(4\pi)} + A_{6\pi}^{(6\pi)}$$

↪ (a) only contributes to $A_{6\pi}^{(6\pi)}$

↪ (b) contributes to both the pole and non-pole parts $A_{6\pi}^{(4\pi)}$ and $A_{6\pi}^{(6\pi)}$

At LO a simple expression

$$A^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{1}{F_\pi^4}\left(2p_1 \cdot p_2 + 2p_3 \cdot p_4 + 2p_5 \cdot p_6 + 3M_\pi^2\right)$$

↪ dependence on momenta is the only one at this order compatible with the symmetries

The main new result is the next-order six-pion subamplitude

↪ split it up into numerous parts:

$$F_\pi^6 A^{(4)}(p_1,p_2,\dots,p_6) = A_{C_3} + A_{C_{21}}^{(1)} + A_{C_{21}}^{(2)} + A_{C_{11}} + A_C^{(1)} + A_C^{(2)} + A_C^{(3)} + A_J^{(1)} + A_J^{(2)} + A_\pi + A_L + A_l$$

↪ each of the terms has the required symmetries under interchange of momenta

Large number of kinematic invariants → reduction to master integrals (scalar triangle integrals) leads to an enormous expression

↪ we have chosen a redundant basis of integrals that have good symmetry properties

Results are rather lengthy, but can be written in a relatively compact way

↪ see paper PRD 104 (2021) 054046, arXiv:2107.06291

We choose a symmetric $3 \to 3$ scattering configuration given by

$$p_1 = \Big(E_p, p, 0, 0\Big) \qquad p_4 = \Big(-E_p, 0, 0, p\Big)$$

$$p_2 = \left(E_p, -\frac{1}{2}p, \frac{\sqrt{3}}{2}p, 0\right) \qquad p_5 = \left(-E_p, \frac{\sqrt{3}}{2}p, 0, -\frac{1}{2}p\right)$$

$$p_3 = \left(E_p, -\frac{1}{2}p, -\frac{\sqrt{3}}{2}p, 0\right) \qquad p_6 = \left(-E_p, -\frac{\sqrt{3}}{2}p, 0, -\frac{1}{2}p\right)$$

We use following numerical inputs:

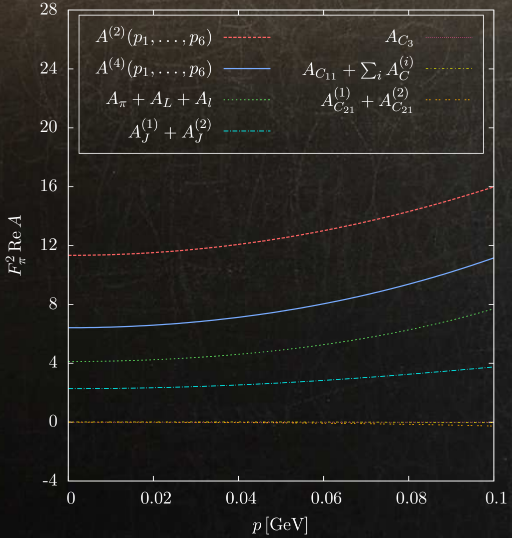
$$M_\pi = 0.139570\,\text{GeV} \qquad \bar{l}_1 = -0.4$$

$$F_\pi = 0.0927\,\text{GeV} \qquad \bar{l}_2 = 4.3$$

$$\mu = 0.77\,\text{GeV} \qquad \bar{l}_3 = 3.41$$

$$N = 3 \qquad \bar{l}_4 = 4.51$$

Bijnens, Ecker, ARNPS 64 (2014)
Colangelo, Gasser, Leutwyler, NPB 603 (2001)
Aoki et al., EPJC 77 (2017)



$F_\pi^2 \mathrm{Re} A$			
$A_{6\pi}^{(4\pi)}$ (LO)	-319.00	$A^{(2)}(p_1,\ldots,p_6)$	15.99
$A_{6\pi}^{(4\pi)}$ (NLO)	-28.54	$A^{(4)}(p_1,\ldots,p_6)$	11.16
$F_\pi^2 \times \mathrm{Re} A / F_\pi^6$			
A_{C_3}	0.002	$A_J^{(1)}$	1.917
$A_{C_{21}}^{(1)}$	-0.948	$A_J^{(2)}$	1.835
$A_{C_{21}}^{(2)}$	0.682	A_π	-2.488
$A_{C_{11}}$	0.090	A_L	8.985
$A_C^{(1)}$	-0.026	A_l	1.209
$A_C^{(2)}$	0.890		
$A_C^{(3)}$	-0.984		

Real parts of the amplitudes for $p = 0.1$ GeV

In the limit $p \to 0$, we find the following analytical expressions:

$$F_\pi^2 A^{(2)}(p_1, p_2, \dots, p_6)\big|_{p\to 0} = 5\,\frac{M_\pi^2}{F_\pi^2} \approx 11.33$$

$$F_\pi^2 \,\mathrm{Re}\, A^{(4)}(p_1, p_2, \dots, p_6)\big|_{p\to 0}$$
$$= \frac{M_\pi^4}{F_\pi^4}\Bigg\{ \underbrace{\frac{1}{18}(-2-225N)\kappa}_{A_\pi} + \underbrace{\frac{1}{6}(-14-75N)L}_{A_L} + \underbrace{(16l_1^r + 56l_2^r + 6l_3^r + 20l_4^r)}_{A_l}$$
$$+ \underbrace{(-44+30N)\kappa}_{A_J^{(1)}} + \underbrace{(24)\kappa}_{A_J^{(2)}} + \frac{1}{2}\kappa\Big[\underbrace{-(30-9N)}_{A_C^{(1)}} + \underbrace{(20)}_{A_C^{(2)}} + \underbrace{(-16)}_{A_C^{(3)}}\Big] \Bigg\} \approx 6.416$$

Summary

We calculated the pion mass, decay constant, the four-pion and six-pion amplitude to NLO in the massive O(N) nonlinear sigma model

$\hookrightarrow$ relevant NLO Lagrangian constructed in analogy with two(-quark)-flavour ChPT Lagrangian
$\hookrightarrow$ our results agree with previous results for $N = 3$ and general-N behaviour

Our main result is the six-pion amplitude
$\hookrightarrow$ split in one-particle reducible and irreducible parts

The reducible part employs the off-shell four-pion amplitude generalizing (beyond $N = 3$) the amplitude given by *Bijnens et al.*, PLB 374 (1996), NPB 508 (1997)

The irreducible part can be divided in a large number of subparts
$\hookrightarrow$ each subpart satisfies the expected permutation symmetries
$\hookrightarrow$ the choice of triangle loop integrals with high symmetry allows for a fairly compact expression
$\hookrightarrow$ NLO correction is sizable but not very large

Outlook

Work in progress
$\hookrightarrow$ combine our results with the methods for extracting three-body scattering from finite volume in lattice QCD

Might be of interest for the amplitude community

More details in PRD 104 (2021) 054046, arXiv:2107.06291

Backup

Massive $O(N+1)/O(N)$ nonlinear sigma model extended beyond the LO

$$\mathcal{L} = \frac{F^2}{2}\,\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi + F^2\chi^\mathsf{T}\Phi + l_1\left(\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi\right)\left(\partial_\nu\Phi^\mathsf{T}\partial^\nu\Phi\right) + l_2\left(\partial_\mu\Phi^\mathsf{T}\partial_\nu\Phi\right)\left(\partial^\mu\Phi^\mathsf{T}\partial^\nu\Phi\right) + l_3\left(\chi^\mathsf{T}\Phi\right)^2 + l_4\partial_\mu\chi^\mathsf{T}\partial^\mu\Phi$$

$$l_i = l_i^\mathrm{r} - \frac{1}{16\pi^2}\frac{\gamma_i}{2}\left(\frac{2}{4-d} - \gamma_\mathsf{E} + \log 4\pi - \log\mu^2 + 1\right), \qquad l_i^r = \frac{1}{16\pi^2}\frac{\gamma_i}{2}\left(\bar{l}_i + \ln\frac{M_\pi^2}{\mu^2}\right)$$

From studying the pion mass, decay constant and the four-pion amplitude

$$\gamma_3 = 1 - \frac{N}{2} \qquad \gamma_1 = \frac{N}{2} - \frac{7}{6}$$

$$\gamma_4 = N - 1 \qquad \gamma_2 = \frac{2}{3}$$