Three-pion scattering and isospin-3 three-particle K-matrix at NLO in ChPT

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Genova, Italy
June 7, 2023

Introduction

In low-energy region, we cannot study perturbatively the interactions of hadrons directly from QCD
$\hookrightarrow$ alternative approaches $\rightarrow$ Chiral perturbation theory (ChPT)

Weinberg, Phys.A 96, (1979), *Gasser and Leutwyler*, Ann.Ph.158 (1984)

Many observables are known in ChPT to a high loop order
$\hookrightarrow$ only recently it has become of interest to calculate the six-pion amplitude at low energies after it has been estimated using lattice QCD

Blanton et al., PRL 124 (2020), JHEP 10 (2021),
Fischer et al., EPJC 81 (2021), *Hansen et al.*, PRL 126 (2021),
Brett et al., PRD 104 (2021)

The six-pion amplitude at tree level was first done using current algebra methods

e.g. *Osborn*, Lett.N.Cim.2 (1969)

It has been redone with Lagrangian methods many times, not known to one-loop order

e.g. *Low et al.*, JHEP 11 (2019), *Bijnens et al.*, JHEP 11 (2019)

We have therefore calculated at NLO the six-pion amplitude,
(as well as the four-pion amplitude, pion mass and decay constant)
$\hookrightarrow$ within ChPT generalization to the O$(N+1)$/O(N) massive nonlinear sigma model
$\hookrightarrow$ two(-quark)-flavour ChPT equivalent to O(4)/O(3)

The relation to the measurement of the lattice is nontrivial to implement given
$\hookrightarrow$ complexity of the three-body finite volume calculations
$\hookrightarrow$ subtraction of the two-body rescatterings involved

Hansen et al., PRD 90 (2014), PRD 92 (2015), *Hammer et al.*, JHEP 09 (2017), JHEP 10 (2017),
Mai et al., EPJA 53 (2017), PRL 122 (2019), *Romero-López et al.*, JHEP 02 (2021),
Blanton et al., PRD 102 (2020)

Theoretical setting
Extended nonlinear sigma model

Massive $\mathrm{O}(N+1)/\mathrm{O}(N)$ nonlinear sigma model extended beyond the LO

$$\mathcal{L} = \frac{F^2}{2}\,\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi + F^2\chi^\mathsf{T}\Phi$$
$$+\,l_1\left(\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi\right)\left(\partial_\nu\Phi^\mathsf{T}\partial^\nu\Phi\right) + l_2\left(\partial_\mu\Phi^\mathsf{T}\partial_\nu\Phi\right)\left(\partial^\mu\Phi^\mathsf{T}\partial^\nu\Phi\right) + l_3\left(\chi^\mathsf{T}\Phi\right)^2 + l_4\partial_\mu\chi^\mathsf{T}\partial^\mu\Phi$$

Φ: real vector of $N+1$ components, $\Phi^\mathsf{T}\Phi = 1$
$\chi^\mathsf{T} = \left(M^2, \vec{0}\right)$

F, M: bare pion decay constant and mass

$\hookrightarrow$ calculate the four-pion and six-pion amplitudes at NLO

External fields can be added as in *Gasser and Leutwyler*, Ann.Ph.158 (1984)

The coefficients (low-energy constants) l_i are free parameters in the theory
$\hookrightarrow$ UV-divergent and -finite parts

$$l_i = l_i^\mathrm{r} - \frac{1}{16\pi^2}\frac{\gamma_i}{2}\left(\frac{2}{4-d} - \gamma_\mathsf{E} + \log 4\pi - \log\mu^2 + 1\right), \qquad l_i^r = \frac{1}{16\pi^2}\frac{\gamma_i}{2}\left(\bar{l}_i + \ln\frac{M_\pi^2}{\mu^2}\right)$$

Theoretical setting

Different parameterizations

$$\Phi_1 = \left(\sqrt{1-\varphi},\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

Gasser and Leutwyler, Ann.Ph.158 (1984)

$$\Phi_2 = \frac{1}{\sqrt{1+\varphi}}\left(1,\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

simple variation

$$\Phi_3 = \left(1-\frac{1}{2}\varphi,\, \sqrt{1-\frac{1}{4}\varphi}\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

ESB term only gives mass terms of ϕ_is

$$\Phi_4 = \left(\cos\sqrt{\varphi},\, \frac{1}{\sqrt{\varphi}}\sin\sqrt{\varphi}\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

follows the general prescription from *Coleman, Wess and Zumino*, PR 177 (1969)

$$\Phi_5 = \frac{1}{1+\frac{1}{4}\varphi}\left(1-\frac{1}{4}\varphi,\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

Weinberg, PR 166 (1968)

$\varphi \equiv \dfrac{\boldsymbol{\phi}^\mathsf{T}\boldsymbol{\phi}}{F^2}$, with $\boldsymbol{\phi}^\mathsf{T} = (\phi_1, \dots, \phi_N)$ a real vector of N components (flavours)

$\hookrightarrow$ few examples of the whole class of parametrizations

$$\Phi = \left(\sqrt{1-\varphi\, f^2(\varphi)},\, f(\varphi)\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T},$$

with $f(x)$ any analytical function satisfying $f(0)=1$

Four-pion amplitude

On-shell amplitude in general

- p_i, $i = 1, \ldots, 4$ pion incoming four-momenta, $\sum p_i = 0$
- f_i flavours

Invariance under rotation in the isospin space and crossing symmetry implies

$$
\begin{aligned}
&A_{4\pi}(p_1, f_1, p_2, f_2, p_3, f_3, f_4) \\
&= \delta_{f_1 f_2}\delta_{f_3 f_4} A(p_1, p_2, p_3) + \delta_{f_1 f_3}\delta_{f_2 f_4} A(p_3, p_1, p_2) + \delta_{f_2 f_3}\delta_{f_1 f_4} A(p_2, p_3, p_1)
\end{aligned}
$$

Mandelstam variables

- $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2$, $s + t + u = 4M^2$
- $\hookrightarrow$ subamplitude $A(p_1, p_2, p_3) = A(s, t, u)$

Up-to-and-including $\mathcal{O}(p^4)$, order by order

$$A(s, t, u) = A^{(2)}(s, t, u) + A^{(4)}(s, t, u)$$

Four-pion amplitude

Leading order

The leading-order $\mathcal{O}(p^2)$ amplitude stems from a single diagram

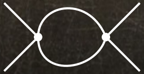
$\hookrightarrow$ schematically $A_{4\pi}^{(2)} = \mathcal{M}_{\text{LO}}^{(2)}\big|_{\text{on-shell}}$

Related LO subamplitude (with LO relations $M \to M_\pi$ and $F \to F_\pi$)

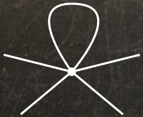
$$A^{(2)}(s,t,u) = \frac{1}{F_\pi^2}\left(s - M_\pi^2\right)$$

Four-pion amplitude
Next-to-leading order

At NLO, one-loop diagrams (two topologies of 4 one-loop diagrams in total) and a counterterm



(a) $3\times$



(b) $1\times$

(c) $1\times$

+ NLO field renormalization, and mass and decay-constant redefinitions applied to the LO graph
↪ schematically $A_{4\pi}^{(4)} = \mathcal{M}_{\text{1-loop}} + \mathcal{M}_{\text{CT}} + 4(Z^{1/2}-1)\mathcal{M}_{\text{LO}}^{(2)} + \mathcal{M}_{\text{LO}}^{(4)}$

The Z factor is related to the pion self-energy Σ

$$\frac{1}{Z} = 1 - \frac{\partial\Sigma(p^2)}{\partial p^2}\bigg|_{p^2=M_\pi^2}$$

Standard relations $M_\pi^2 = M^2 + \overline{\Sigma}$, $F_\pi = F(1+\delta F)$ give the substitutions at the given order

$$M^k \to M_\pi^k\left(1 - \frac{k}{2}\frac{\overline{\Sigma}}{M_\pi^2}\right), \qquad \overline{\Sigma} = \frac{M_\pi^4}{F_\pi^2}\left[2l_3^{\mathrm{r}} + \frac{1}{2}(N-2)L\right] + \mathcal{O}\left(\frac{1}{F_\pi^4}\right)$$

$$\frac{1}{F^k} \to \frac{1}{F_\pi^k}(1+k\delta F), \qquad \delta F = \frac{M_\pi^2}{F_\pi^2}\left[l_4^{\mathrm{r}} - \frac{1}{2}(N-1)L\right] + \mathcal{O}\left(\frac{1}{F_\pi^4}\right)$$

Four-pion amplitude
Next-to-leading-order result

Parametrization-independent and UV-finite result

$$
\begin{aligned}
F_\pi^4 A^{(4)}(s,t,u) = (t-u)^2 \left(-\frac{5}{36}\kappa - \frac{1}{6}L + \frac{1}{2}l_2^{\mathrm{r}}\right) \\
+ M_\pi^2 s\left[\left(N-\frac{29}{9}\right)\kappa + \left(N-\frac{11}{3}\right)L - 8l_1^{\mathrm{r}} + 2l_4^{\mathrm{r}}\right] \\
+ s^2\left[\left(\frac{11}{12}-\frac{N}{2}\right)\kappa + \left(1-\frac{N}{2}\right)L + 2l_1^{\mathrm{r}} + \frac{1}{2}l_2^{\mathrm{r}}\right] \\
+ M_\pi^4\left[\left(\frac{20}{9}-\frac{N}{2}\right)\kappa + \left(\frac{8}{3}-\frac{N}{2}\right)L + 8\,l_1^{\mathrm{r}} + 2l_3^{\mathrm{r}} - 2l_4^{\mathrm{r}}\right] \\
+ \bar{J}(s)\left[\left(\frac{N}{2}-1\right)s^2 + (3-N)M_\pi^2 s + \left(\frac{N}{2}-2\right)M_\pi^4\right] \\
+ \left\{\frac{1}{6}\bar{J}(t)\left[2t^2 - 10M_\pi^2 t - 4M_\pi^2 s + st + 14M_\pi^4\right] + (t\leftrightarrow u)\right\}
\end{aligned}
$$

Above we used

$$
\kappa = \frac{1}{16\pi^2}\,, \quad L \equiv \kappa\log\frac{M_\pi^2}{\mu^2}\,, \quad \bar{J}(q^2) \equiv \kappa\left(2+\sigma\log\frac{\sigma-1}{\sigma+1}\right), \quad \sigma = \sqrt{1-\frac{4M_\pi^2}{q^2}}
$$

Form as given in *Bijnens et al.*, PLB 374 (1996), NPB 508 (1997), generalized to $N \neq 3$

↪ somewhat different from the form given in *Gasser and Leutwyler*, Ann.Ph.158 (1984)

↪ equivalent to a given order but different off-shell extrapolations

The expressions agree with the known results

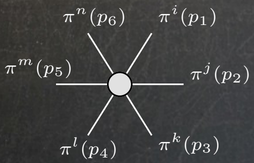
↪ for $N=3$, *Gasser and Leutwyler*, Ann.Ph.158 (1984)

↪ on the N dependence, e.g. *Dobado and Morales*, PRD 52 (1995), *Bijnens and Carloni*, NPB 827 (2010), NPB 843 (2011)

Six-pion amplitude

4 pions $\longrightarrow$ 3 channels/permutations/ways to distribute 4 pions in 2 pairs

6 pions $\longrightarrow$ 10 ways in 2 groups of three (P_{10})
$\hookrightarrow$ 15 ways in 3 pairs (P_{15})



The full six-pion amplitude at $\mathcal{O}(p^4)$

$$A_{6\pi} = A_{6\pi}^{(\text{pole})} + A_{6\pi}^{(\text{non-pole})}$$

$\hookrightarrow$ (a) only contributes to $A_{6\pi}^{(\text{non-pole})}$
$\hookrightarrow$ (b) contributes to both the pole and non-pole parts

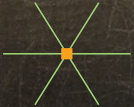
$A_{6\pi}^{(\text{pole})}$ can be written in terms of the four-pion amplitude and $A_{6\pi}^{(\text{non-pole})}$ is the remainder

(a) $1\times$

(b) $10\times$

Six-pion amplitude

Feynman diagrams of relevant topologies

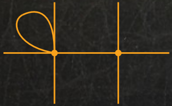


(a) $1\times$

(b) $1\times$

(c) $15\times$

(d) $20\times$



(e) $20\times$

(f) $60\times$

(g) $10\times$

(h) $10\times$

(i) $15\times$

Six-pion amplitude

The pole part

$$A_{6\pi}^{(\text{pole})} \equiv \sum_{P_{10}, f_o} A_{4\pi}(p_i, f_i, p_j, f_j, p_k, f_k, f_o) \frac{(-1)}{p_{ijk}^2 - M_\pi^2} A_{4\pi}(p_l, b_l, p_m, f_m, p_n, f_n, f_o)$$

$\hookrightarrow$ residue at the pole unique, off-shell extrapolation away from $p_{ijk}^2 \equiv (p_i + p_j + p_k)^2 = M_\pi^2$ not

$A_{4\pi}(p_i, f_i, p_j, f_j, p_k, f_k, f_o)$ is the four-pion amplitude with one leg off-shell

$$\begin{aligned} &A_{4\pi}(p_i, f_i, p_j, f_j, p_k, f_k, f_o) \\ &= \delta_{f_i f_j} \delta_{f_k f_o} A(p_i, p_j, p_k) + \delta_{f_i f_k} \delta_{f_j f_o} A(p_k, p_i, p_j) + \delta_{f_j f_k} \delta_{f_i f_o} A(p_j, p_k, p_i) \end{aligned}$$

The (four-pion) subamplitude $A(p_i, p_j, p_k) = A(s, t, u)$ is defined as usual

$\hookrightarrow$ $s = (p_i + p_j)^2$, $t = (p_i + p_k)^2$ and $u = (p_j + p_k)^2$, although now $s + t + u = 3M_\pi^2 + p_{ijk}^2$

We have chosen a particular form for the off-shell four-pion subamplitude $A(s, t, u)$

$\hookrightarrow$ other off-shell extrapolations are possible and will lead to a different $A_{6\pi}^{(\text{non-pole})}$

$\hookrightarrow$ independent of the parametrization used

- $\hookrightarrow$ also $A_{4\pi}$ and, consequently, the respective parts $A_{6\pi}^{(\text{pole})}$ and $A_{6\pi}^{(\text{non-pole})}$ by definition

$\hookrightarrow$ the way how the contributions from the one-particle irreducible and reducible diagrams are distributed within the final result is parametrization dependent

Six-pion amplitude
The non-pole part

$$A_{6\pi}^{\text{(non-pole)}} \equiv \sum_{P_{15}} \delta_{f_i f_j} \delta_{f_k f_l} \delta_{f_m f_n} A(p_i, p_j, p_k, p_l, p_m, p_n)$$

The (six-pion) subamplitude $A(p_1, p_2, p_3, p_4, p_5, p_6)$

$\hookrightarrow$ no poles, only cuts (however, the imaginary part of the triangle integrals can contain poles)

$\hookrightarrow$ function of three pairs of momenta

- $\hookrightarrow$ fully symmetric under the interchange of any of the pairs
- $\hookrightarrow$ symmetric for the interchange within a pair

The six-pion subamplitude respecting orders in the expansion

$$A(p_1, p_2, p_3, p_4, p_5, p_6) = A^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) + A^{(4)}(p_1, p_2, p_3, p_4, p_5, p_6)$$

Six-pion amplitude
Non-pole part at LO

At LO a simple expression

$$A^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{1}{F_\pi^4}\left(2p_1 \cdot p_2 + 2p_3 \cdot p_4 + 2p_5 \cdot p_6 + 3M_\pi^2\right)$$

$\hookrightarrow$ dependence on momenta is the only one at this order compatible with the symmetries

(a) $1\times$

(b) $10\times$

The main new result is the next-order six-pion subamplitude

↪ split it up into numerous parts:

$$F_\pi^6 A^{(4)}(p_1, p_2, \ldots, p_6) = A_{C_3} + A_{C_{21}}^{(1)} + A_{C_{21}}^{(2)} + A_{C_{11}} + A_C^{(1)} + A_C^{(2)} + A_C^{(3)} + A_J^{(1)} + A_J^{(2)} + A_\pi + A_L + A_l$$

↪ each of the terms has the required symmetries under interchange of momenta

Large number of kinematic invariants → reduction to master integrals (scalar triangle integrals) leads to an enormous expression

↪ we have chosen a redundant basis of integrals that have good symmetry properties

Results are rather lengthy, but can be written in a relatively compact way

↪ see paper *Bijnens and TH*, PRD 104 (2021) 054046, arXiv:2107.06291

Six-meson amplitude in QCD-like theories

$\hookrightarrow$ with global symmetry and breaking patterns
$\mathrm{SU}(n) \times \mathrm{SU}(n)/\,\mathrm{SU}(n)$ (extending the QCD case), $\mathrm{SU}(2n)/\,\mathrm{SO}(2n)$, and $\mathrm{SU}(2n)/\,\mathrm{Sp}(2n)$

Relevant Lagrangian for meson–meson scattering at NLO, $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)}$

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle \\
\mathcal{L}^{(4)} &= L_0\langle u_\mu u_\nu u^\mu u^\nu\rangle + L_1\langle u_\mu u^\mu\rangle\langle u_\nu u^\nu\rangle \\
&+ L_2\langle u_\mu u_\nu\rangle\langle u^\mu u^\nu\rangle + L_3\langle u_\mu u^\mu u_\nu u^\nu\rangle \\
&+ L_4\langle u_\mu u^\mu\rangle\langle \chi_+\rangle + L_5\langle u_\mu u^\mu \chi_+\rangle \\
&+ L_6\langle \chi_+\rangle^2 + L_7\langle \chi_-\rangle^2 + \tfrac{1}{2}L_8\langle \chi_+^2 + \chi_-^2\rangle
\end{aligned}$$

$$u_\mu \equiv i(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger)$$

$$\chi_\pm \equiv u^\dagger \chi u^\dagger \pm u\chi^\dagger u \qquad\qquad u = \exp\left(\frac{i}{\sqrt{2}F}\,\phi^a t^a\right)$$

Bijnens, TH and Sjö, Phys. Rev. D 106 (2022) 054021, arXiv:2206.14212

Lattice QCD

↪ provides systematically improvable approach to calculate strongly-interacting processes

↪ also processes inaccessible in experiment

↪ ex.: $3\pi^+ \to 3\pi^+$

↪ can treat strong force in isolation (absence of weak and electromagnetic interactions)

↪ lightest mesons and some other hadrons stable

↪ three-particle processes recently received a lot of attention

↪ formalism developed over last decade

↪ 3-particle finite-volume spectrum computed on lattice $\to$ amplitudes

↪ several main approaches

The three-particle RFT formalism and the role of $\mathcal{K}_{\mathrm{df},3}$

Hansen and Sharpe, PRD 90 (2014)

$\hookrightarrow$ short-range three-body interaction parametrized via intermediate cutoff-dependent quantity

$\hookrightarrow$ three-particle K-matrix $\mathcal{K}_{\mathrm{df},3}$

$\mathcal{K}_{\mathrm{df},3}$ real, smooth, invariant under the same symmetries as $\mathcal{M}_3$

$\hookrightarrow$ unitarity branch cuts (2- and 3-particle) and 1-particle pole (OPE) absent by construction

$\hookrightarrow$ good check of derivation from NLO ChPT

$\hookrightarrow$ $\mathcal{K}_{\mathrm{df},3}$ smooth $\to$ can be expanded about threshold

Central equation: quantization condition

$\hookrightarrow$ solutions correspond to the energy levels E_n of three-pion system

$\hookrightarrow$ total three-momentum $\boldsymbol{P}$ in a box of side L

$$\det\left[F_3^{-1}(E,\boldsymbol{P},L)+\mathcal{K}_{\mathrm{df},3}(E^*)\right]=0 \qquad \text{at} \qquad E=E_n$$

(valid in the energy range where only three-pion intermediate states can go on shell)

F_3 depends on the volume, kinematic functions and two-particle interactions

Threshold expansion of $\mathcal{K}_{\mathrm{df},3}$

$\Longrightarrow$ given a set of $3\pi^+$ finite-volume energy levels $\to$ extract $\mathcal{K}_{\mathrm{df},3}$ by fitting
$\hookrightarrow$ parametrization of $\mathcal{K}_{\mathrm{df},3}$ in terms of few independent quantities

Expand $\mathcal{K}_{\mathrm{df},3}$ in terms of relativistic invariants $\leftrightarrow$ distance to the 3-particle threshold
$\hookrightarrow$ $\mathcal{K}_{\mathrm{df},3}$ has symmetries as the scattering amplitude (parity, time-reversal, particle-exchange)
$\hookrightarrow$ only five independent terms contribute to the expansion up to quadratic order:

$$M_\pi^2 \mathcal{K}_{\mathrm{df},3} = \underbrace{\mathcal{K}_0 + \mathcal{K}_1 \Delta}_{\mathrm{LO}} + \mathcal{K}_2 \Delta^2 + \underbrace{\mathcal{K}_{\mathrm{A}} \Delta_{\mathrm{A}} + \mathcal{K}_{\mathrm{B}} \Delta_{\mathrm{B}}}_{\text{angular dependence}} + \mathcal{O}(\Delta^3)$$

$$\tilde{t}_{ij} \equiv \frac{(p_i - k_j)^2}{9M_\pi^2}$$

$$\Delta_j \equiv \sum_i \tilde{t}_{ij} + \Delta = \frac{(P-k_j)^2 - 4M_\pi^2}{9M_\pi^2}\,, \quad \Delta'_i \equiv \sum_j \tilde{t}_{ij} + \Delta = \frac{(P-p_i)^2 - 4M_\pi^2}{9M_\pi^2}$$

$$\Delta \equiv -\frac{1}{2} \sum_{i,j} \tilde{t}_{ij} = \frac{P^2 - 9M_\pi^2}{9M_\pi^2}\,, \quad \Delta_{\mathrm{A}} \equiv \sum_i \left(\Delta_i^2 + \Delta_i'^2\right) - \Delta^2\,, \quad \Delta_{\mathrm{B}} \equiv \sum_{i,j} \tilde{t}_{ij}^2 - \Delta^2$$

k_1, k_2, k_3 incoming, p_1, p_2, p_3 outgoing, i.e. $P = k_1 + k_2 + k_3 = p_1 + p_2 + p_3$

Discrepancy with LO ChPT

In general not obvious how to connect $\mathcal{M}_3$ to $\mathcal{K}_{\mathrm{df},3}$
$\hookrightarrow$ relation given by integral equation which needs to be inverted

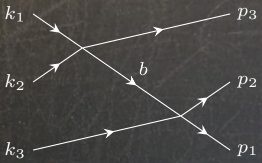
LO ChPT prediction for $\mathcal{K}_{\mathrm{df},3}$ for $3\pi^+$ system determined in *Blanton et al.*, PRL 124 (2020)
$\hookrightarrow$ significant disagreement observed
- $\hookrightarrow$ surprising since its two-particle counterpart described well by LO ChPT
- $\hookrightarrow$ *Blanton et al.*, JHEP 10 (2021)
  - $\hookrightarrow$ different coefficients of the threshold expansion at three values of the pion mass

possible sources of discrepancy
$\hookrightarrow$ systematic errors in lattice calculation (numerically challenging)
$\hookrightarrow$ importance of higher-order ChPT corrections

At LO,

$$\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO}} = \mathcal{M}_{\mathrm{df},3}^{\mathrm{LO}}\,, \qquad \mathcal{M}_{\mathrm{df},3}^{\mathrm{LO}} = \mathcal{M}_3^{\mathrm{LO}} - \mathcal{S}\left\{\mathcal{D}^{(u,u)\mathrm{LO}}(\boldsymbol{p},\boldsymbol{k})\right\}$$

$$\mathcal{D}^{(u,u)\mathrm{LO}}(\boldsymbol{p},\boldsymbol{k}) = -\mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{p})G^{\infty}(\boldsymbol{p},\boldsymbol{k})\mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{k})$$



At NLO, simplified version of similar type:

$$\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}} = \mathrm{Re}\,\mathcal{M}_{\mathrm{df},3}^{\mathrm{NLO}}\,, \qquad \mathcal{M}_{\mathrm{df},3}^{\mathrm{NLO}} = \mathcal{M}_3^{\mathrm{NLO}} - \mathcal{S}\left\{\mathcal{D}^{(u,u)\mathrm{NLO}}(\boldsymbol{p},\boldsymbol{k})\right\}$$

$$\begin{aligned}\mathcal{D}^{(u,u)\mathrm{NLO}}(\boldsymbol{p},\boldsymbol{k}) = &-\mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{p})G^{\infty}(\boldsymbol{p},\boldsymbol{k})\mathcal{M}_2^{\mathrm{NLO}}(\boldsymbol{k}) - \mathcal{M}_2^{\mathrm{NLO}}(\boldsymbol{p})G^{\infty}(\boldsymbol{p},\boldsymbol{k})\mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{k})\\ &+\int_r \mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{p})G_{ss}^{\infty}(\boldsymbol{p},\boldsymbol{r})\mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{r})G_{ss}^{\infty}(\boldsymbol{r},\boldsymbol{k})\mathcal{M}_2^{\mathrm{LO}}(\boldsymbol{k})\end{aligned}$$

$\Longrightarrow$ both at LO and NLO, it turns out that relation $\mathcal{M}_3 \leftrightarrow \mathcal{K}_{\mathrm{df},3}$ is algebraic and linear

$$G^{\infty}(\boldsymbol{p},\boldsymbol{k})_{\ell'm';\ell m} = \left(\frac{k_p^*}{q_{2,p}^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{\boldsymbol{k}}_p^*)H(x_p)H(x_k)Y_{\ell m}^*(\hat{\boldsymbol{p}}_k^*)}{b_{pk}^2 - M_\pi^2 + i\epsilon}\left(\frac{p_k^*}{q_{2,k}^*}\right)^{\ell}$$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35 + 12\log 3) - \mathcal{D}_0 + 111L + \ell^{\mathrm{r}}_{(0)}\right]$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}(1999 + 1920\log 3) - \mathcal{D}_1 + 384L + \ell^{\mathrm{r}}_{(1)}\right]$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923 - 420\log 3) - \mathcal{D}_2 + 360L + \ell^{\mathrm{r}}_{(2)}\right]$$

$$\mathcal{K}_{\mathrm{A}} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809 - 1050\log 3) - \mathcal{D}_{\mathrm{A}} - 9L + \ell^{\mathrm{r}}_{(\mathrm{A})}\right]$$

$$\mathcal{K}_{\mathrm{B}} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698 - 245\log 3) - \mathcal{D}_{\mathrm{B}} + 54L + \ell^{\mathrm{r}}_{(\mathrm{B})}\right]$$

$$\kappa \equiv \frac{1}{16\pi^2}, \quad L \equiv \kappa \log \frac{M_\pi^2}{\mu^2}$$

$$\mathcal{D}_0 \approx -0.0563476589 \qquad \ell^{\mathrm{r}}_{(0)} = -288\ell^{\mathrm{r}}_1 - 432\ell^{\mathrm{r}}_2 - 36\ell^{\mathrm{r}}_3 + 72\ell^{\mathrm{r}}_4$$

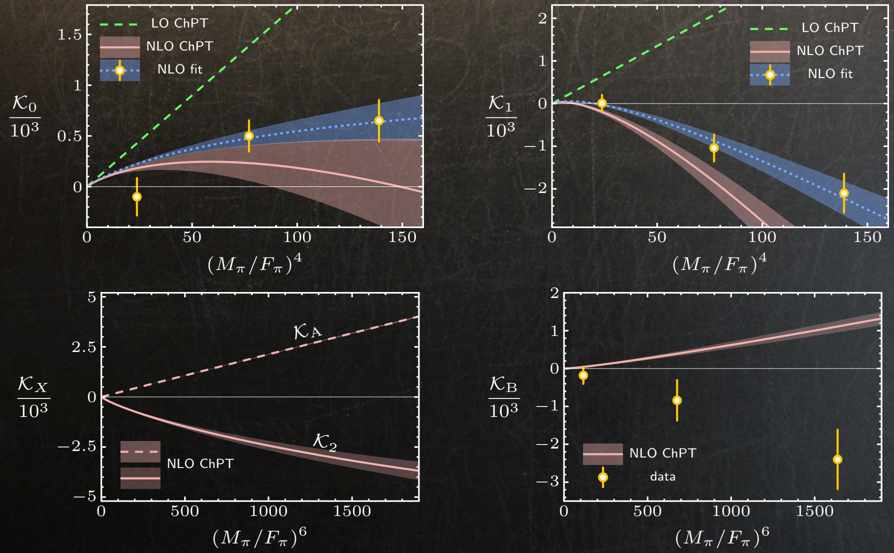
$$\mathcal{D}_1 \approx 0.129589681 \qquad \ell^{\mathrm{r}}_{(1)} = -612\ell^{\mathrm{r}}_1 - 1170\ell^{\mathrm{r}}_2 + 108\ell^{\mathrm{r}}_4$$

$$\mathcal{D}_2 \approx 0.432202370 \qquad \ell^{\mathrm{r}}_{(2)} = -432\ell^{\mathrm{r}}_1 - 864\ell^{\mathrm{r}}_2$$

$$\mathcal{D}_{\mathrm{A}} \approx 9.07273890 \times 10^{-4} \qquad \ell^{\mathrm{r}}_{(\mathrm{A})} = 27\ell^{\mathrm{r}}_1 + \frac{27}{2}\ell^{\mathrm{r}}_2$$

$$\mathcal{D}_{\mathrm{B}} \approx 1.62394747 \times 10^{-4} \qquad \ell^{\mathrm{r}}_{(\mathrm{B})} = -162\ell^{\mathrm{r}}_1 - 81\ell^{\mathrm{r}}_2$$

Comparison to lattice results from *Blanton et al.*, JHEP 10 (2021)



Summary

Six-pion amplitude calculated to NLO in the massive O(N) nonlinear sigma model

$\hookrightarrow$ OPE part employs the off-shell four-pion amplitude generalizing (beyond $N = 3$) the amplitude given by *Bijnens et al.*, PLB 374 (1996), NPB 508 (1997)

$\hookrightarrow$ non-OPE part divided in a large number of subparts
  $\hookrightarrow$ NLO correction turns out to be sizable

$\hookrightarrow$ more details in PRD 104 (2021) 054046, arXiv:2107.06291

Six-meson amplitude in QCD-like theories
$\hookrightarrow$ PRD 106 (2022) 054021, arXiv:2206.14212

Combined with the methods for extracting three-body scattering from finite volume (lattice)
$\hookrightarrow$ JHEP 05 (2023) 187, arXiv:2303.13206
$\hookrightarrow$ analytic results for the threshold expansion for almost all parts
$\hookrightarrow$ agreement in $\mathcal{K}_0$ and $\mathcal{K}_1$ significantly improved with NLO effects, $\mathcal{K}_\mathrm{B}$ new challenge

Outlook

Work in progress: $I < 3$

Mixed-mesons systems at maximum isospin ($K^+\pi^+\pi^+$, $K^+K^+\pi^+$) available from lattice
$\hookrightarrow$ tension with LO ChPT, *Draper et al.*, JHEP 05 (2023) 137

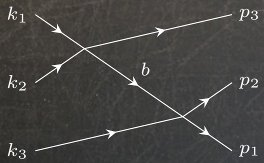
Results on six-meson amplitudes might be of interest to the amplitude community

SPARES

Divergence-free three-particle amplitude $\mathcal{M}_{\mathrm{df},3}$

$$\mathcal{M}_{\mathrm{df},3} = \mathcal{M}_3 - \mathcal{S}\Big\{\mathcal{D}^{(u,u)}(\boldsymbol{p},\boldsymbol{k})\Big\}$$

$\mathcal{S}$ indicates symmetrization
[over choices of initial and final spectators (9 terms in total)]



$\mathcal{D}^{(u,u)}$ is the unsymmetrized subtraction term

↪ numerically challenging

↪ integral equation that can be expanded in powers of $\mathcal{M}_2$

$$\mathcal{D}^{(u,u)}(\boldsymbol{p},\boldsymbol{k}) = -\mathcal{M}_2(\boldsymbol{p})G^\infty(\boldsymbol{p},\boldsymbol{k})\mathcal{M}_2(\boldsymbol{k}) + \int_r \mathcal{M}_2(\boldsymbol{p})G^\infty(\boldsymbol{p},\boldsymbol{r})\mathcal{M}_2(\boldsymbol{r})G^\infty(\boldsymbol{r},\boldsymbol{k})\mathcal{M}_2(\boldsymbol{k}) + \dots$$

$$G^\infty(\boldsymbol{p},\boldsymbol{k})_{\ell'm';\ell m} = \left(\frac{k_p^*}{q_{2,p}^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{\boldsymbol{k}}_p^*)H(x_p)H(x_k)Y_{\ell m}^*(\hat{\boldsymbol{p}}_k^*)}{b_{pk}^2 - M_\pi^2 + i\epsilon}\left(\frac{p_k^*}{q_{2,k}^*}\right)^{\ell}$$

$$\int_r \equiv \int \frac{\mathrm{d}^3 r}{(2\pi)^3 2\omega_r}, \quad \omega_r = \sqrt{\boldsymbol{r}^2 + M_\pi^2}$$

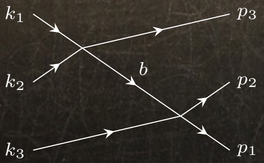
$b_{pk} \equiv P - p - k$, $x_k \equiv (P-k)^2/(4M_\pi^2)$ and similarly for x_p

$H(x)$ is a smooth cutoff function that is 0 when $x \leq 0$ and 1 when $x \geq 1$

$$H(x) = \exp\Big[-\tfrac{1}{x}\,\exp\big(-\tfrac{1}{1-x}\big)\Big], \quad 0 < x < 1$$

Explicit calculation of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO}}$ for $3\pi^+$ scattering

OPE



$$\begin{aligned}\mathcal{M}_{2,\mathrm{off}}^{\mathrm{LO}}(\boldsymbol{p}_3) &= A^{(2)}(t_2,u_2,s_2) + A^{(2)}(u_2,s_2,t_2)\\ &= \frac{1}{F_\pi^2}(t_2+u_2-2M_\pi^2)\\ &= \frac{1}{F_\pi^2}\left[-2p_1\cdot p_2 + (b^2-M_\pi^2)\right]\end{aligned}$$

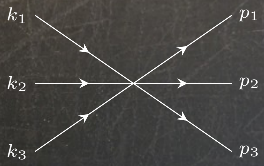
$$\begin{aligned}&\mathcal{K}_{\mathrm{df},3,s}^{(u,u)\mathrm{LO,OPE}}(\boldsymbol{p}_3,\boldsymbol{k}_3)\\ &\quad= -\mathcal{M}_{2,\mathrm{off}}^{\mathrm{LO}}(\boldsymbol{p}_3)\,\frac{1}{b^2-M_\pi^2+i\epsilon}\,\mathcal{M}_{2,\mathrm{off}}^{\mathrm{LO}}(\boldsymbol{k}_3) + \mathcal{M}_{2s}^{\mathrm{LO}}(\boldsymbol{p}_3)G_{ss}^{\infty}(\boldsymbol{p}_3,\boldsymbol{k}_3)\mathcal{M}_{2s}^{\mathrm{LO}}(\boldsymbol{k}_3)\\ &\quad= \frac{1}{F_\pi^4}\Big[2p_1\cdot p_2 + 2k_1\cdot k_2 - (b^2-M_\pi^2)\Big]\end{aligned}$$

$\Downarrow$ (symmetrization)

$$F_\pi^4\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO,OPE}} = 7P^2 - 27M_\pi^2$$

Explicit calculation of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO}}$ for $3\pi^+$ scattering

non-OPE



$$A_{6\pi}^{(\text{non-pole})} \equiv \sum_{P_{15}} \delta_{f_i f_j} \delta_{f_k f_l} \delta_{f_m f_n} A(p_i, p_j, p_k, p_l, p_m, p_n)$$

$$\Downarrow (I = 3)$$

$$\begin{aligned}
\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO,non\text{-}OPE}} &= A^{(2)}(k_1, -p_1, k_2, -p_2, k_3, -p_3) + A^{(2)}(k_1, -p_2, k_2, -p_1, k_3, -p_3) \\
&+ A^{(2)}(k_1, -p_1, k_2, -p_3, k_3, -p_2) + A^{(2)}(k_1, -p_2, k_2, -p_3, k_3, -p_1) \\
&+ A^{(2)}(k_1, -p_3, k_2, -p_1, k_3, -p_2) + A^{(2)}(k_1, -p_3, k_2, -p_2, k_3, -p_1) \\
&= \left| F_\pi^4 A^{(2)}(k_1, -p_1, k_2, -p_2, k_3, -p_3) = -2k_1 \cdot p_1 - 2k_2 \cdot p_2 - 2k_3 \cdot p_3 + 3M_\pi^2 \right| \\
&= \frac{1}{F_\pi^4} \left[-4P^2 + 18M_\pi^2 \right]
\end{aligned}$$

$$F_\pi^4 \mathcal{K}_{\mathrm{df},3}^{\mathrm{LO}} = F_\pi^4 \left(\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO,OPE}} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{LO,non\text{-}OPE}} \right) = 3P^2 - 9M_\pi^2 = \left| P^2 = 9M_\pi^2(1+\Delta) \right| = M_\pi^2(18 + 27\Delta)$$

Bull's head subtraction

$$\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}} = \mathrm{Re}\,\mathcal{M}_{\mathrm{df},3}^{\mathrm{NLO}} = \mathrm{Re}\,\mathcal{M}_3^{\mathrm{NLO,non\text{-}OPE}} - \mathrm{Re}\,\mathcal{D}^{\mathrm{BH}} + \mathrm{Re}\Big\{\mathcal{M}_3^{\mathrm{NLO,OPE}} - \mathcal{D}^{\mathrm{NLO,OPE}}\Big\}$$

$$\mathcal{D}^{(u,u)\mathrm{BH}}(\boldsymbol{p}_3,\boldsymbol{k}_3) = -\frac{1}{F_\pi^6}(2p_1\cdot p_2)\, I(\boldsymbol{p}_3,\boldsymbol{k}_3)(2k_1\cdot k_2)$$

$$I(\boldsymbol{p}_3,\boldsymbol{k}_3) \equiv \int_r \frac{H(x_r)\big[(P-r)^2 - 2M_\pi^2\big]H(x_r)}{\big[(p_+-r)^2 - M_\pi^2 + i\epsilon\big]\big[(k_+-r)^2 - M_\pi^2 + i\epsilon\big]}$$

$$\Downarrow$$

$$H_{m,n} \equiv \int_0^{1/\sqrt{3}} \mathrm{d}z\, \underbrace{\frac{1}{\pi^2}\frac{\sqrt{1+z^2}}{z^m}}_{\frac{\mathrm{d}}{\mathrm{d}z}f_m(z)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} H^2(x)$$

$$\Downarrow$$

$$\frac{F_\pi^6}{M_\pi^4}\mathrm{Re}\,\mathcal{D}^{\mathrm{BH}} = \Big[96\kappa + 9f_0 + \mathcal{D}_0\Big] + \Delta\big[296\kappa + 24f_0 + \mathcal{D}_1\big] + \Delta^2\big[\tfrac{5661}{50}\kappa + \tfrac{621}{40}f_0 + \mathcal{D}_2\big]$$
$$+ \Delta_{\mathrm{A}}\big[-\tfrac{1764}{5}\kappa + \tfrac{135}{32}f_0 + \mathcal{D}_{\mathrm{A}}\big] + \Delta_{\mathrm{B}}\big[-\tfrac{612}{25}\kappa + \tfrac{189}{160}f_0 + \mathcal{D}_{\mathrm{B}}\big]$$

$$\big[f_0 \equiv f_0(1/\sqrt{3}) = \tfrac{4}{3}\kappa(4 + 3\log 3)\big]$$

Contributions to $\mathcal{K}_{\mathrm{df},3}$

Numerical comparison

	$\left(\frac{F_\pi}{M_\pi}\right)^6\mathcal{K}_0$	$\left(\frac{F_\pi}{M_\pi}\right)^6\mathcal{K}_1$	$\left(\frac{F_\pi}{M_\pi}\right)^6\mathcal{K}_2$	$\left(\frac{F_\pi}{M_\pi}\right)^6\mathcal{K}_\mathrm{A}$	$\left(\frac{F_\pi}{M_\pi}\right)^6\mathcal{K}_\mathrm{B}$
non-OPE	−2.04(28)	−3.75(61)	1.43(37)	3.00(14)	0.25(28)
OPE	0.50(53)	−1.8(1.0)	−5.11(58)	−2.76(15)	−0.22(37)
BH, excl. $\mathcal{D}_X$	−1.16234	−3.35289	−1.67334	1.97425	0.08225
BH, only $\mathcal{D}_X$	0.05635	−0.12959	−0.43220	−0.00091	−0.00016
Total NLO	−2.65(26)	−9.04(46)	−5.79(24)	2.212(16)	0.118(93)

	$\mathcal{K}_0$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_\mathrm{A}$	$\mathcal{K}_\mathrm{B}$
LO	94.5186	141.778	0	0	0
NLO	−31.9(3.1)	−108.8(5.5)	−69.6(2.9)	26.62(19)	1.4(1.1)
Total	62.6(3.1)	34.0(5.5)	−69.6(2.9)	26.62(19)	1.4(1.1)