Dynamical Generation of Fermion Mixing

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Contents

Motivations

- Flavor mixing is a central ingredient in the Standard Model;
- Mixing transformations for fields have been shown to be non–trivial since they induce a condensate structure in the vacuum[1];
- This suggests the idea of dynamical generation of mixing in a similar way as it happens for the masses.

[1] M.Blasone, G.Vitiello (1995).

Neutrino Mixing in QFT

Massive neutrinos

Mixing transformations defines fields with definite mass ν_1, ν_2:

$$\begin{aligned}\nu_e(x) &= \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)\\ \nu_\mu(x) &= -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)\end{aligned}$$

with

$$\tan 2\theta = \frac{2m_{e\mu}}{m_e - m_\mu}$$

These relations define neutrinos with definite mass.

Mass eigenstates

Free fields with definite masses can be expanded as:

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}(t) \alpha^r_{\mathbf{k},i} \; + \; v^r_{-\mathbf{k},i}(t) \beta^{r\dagger}_{-\mathbf{k},i} \right] e^{i\mathbf{k}\cdot\mathbf{x}} \,, \quad i = 1,2$$

A mass-eigenstate neutrino is defined as:

$$|\nu^r_{\mathbf{k},i}\rangle = \alpha^{r\dagger}_{\mathbf{k},i} |0\rangle_{1\,2} \,, \quad i = 1,2$$

where the *mass* vacuum satisfies:

$$\alpha^r_{\mathbf{k},i} |0\rangle_{1\,2} \; = \; \beta^r_{\mathbf{k},i} |0\rangle_{1\,2} \,, \quad i = 1,2$$

Flavor Charges

Flavor charges[2]:

$$Q_{\nu_e}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\nu_e(x)\,, \qquad Q_{\nu_\mu}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\nu_\mu(x)$$

Total flavor charge:

$$Q \;=\; Q_{\nu_e}(t) \;+\; Q_{\nu_\mu}(t)$$

Because of mixing, only this last is conserved.

[2] M.Blasone, P.Jizba, G.Vitiello (2001)

Standard Flavor eigenstates

Flavor eigenstates, are usually taken as a simple combination of mass eigenstates:

$$
\begin{aligned}
|\nu^r_{\mathbf{k},e}\rangle_P &= \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle \\
|\nu^r_{\mathbf{k},\mu}\rangle_P &= -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle
\end{aligned}
$$

ACHTUNG! These are NOT eigenstates of the flavor charges:

$$
\begin{aligned}
{}_P\langle\nu^r_{\mathbf{k},e}| :Q_{\nu_e}(0): |\nu^r_{\mathbf{k},e}\rangle_P &= \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta \;<\; 1 \\
{}_P\langle\nu^r_{\mathbf{k},e}| :Q_{\nu_\mu}(0): |\nu^r_{\mathbf{k},e}\rangle_P &= 2\,(1-|U_{\mathbf{k}}|)\sin^2\theta\cos^2\theta \;>\; 0
\end{aligned}
$$

with $U_{\mathbf{k}} = u^{r\dagger}_{2,\mathbf{k}} u^r_{1,\mathbf{k}}$.

Mixing generator

In a finite volume, mixing relations are rewritten

$$\begin{aligned} \nu_e^\alpha(x) &= G_\theta^{-1}(t)\nu_1^\alpha(x)\, G_\theta(t) \\ \nu_\mu^\alpha(x) &= G_\theta^{-1}(t)\, \nu_2^\alpha(x) G_\theta(t) \end{aligned}$$

where mixing generator has been introduced

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\, \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

Decomposition of the mixing generator

The mixing generator can be decomposed as[3]:

$$G_\theta \;=\; B(\Theta_1,\Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1,\Theta_2)$$

where

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},2}\right) e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^{r}_{-\mathbf{k},1}\right) e^{-i\psi_k}\right]\right\}$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\; \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i} e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i} e^{i\phi_{k,i}}\right]\right\}, i = 1,2$$

and $B(\vartheta_1,\vartheta_2) \equiv B_1(\vartheta_1)\, B_2(\vartheta_2)$.

[3] M.Blasone, M.V.Gargiulo, G.Vitiello (2015)

$B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are Bogoliubov transformations which induces a mass shift and $R(\theta)$ is a rotation.

Their action on the mass vacuum is:

$$
\begin{aligned}
|\widetilde{0}\rangle_{1,2} &\equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \\
&= \prod_{\mathbf{k},r} \Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i} \Big] |0\rangle_{1,2} \\
& \quad R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}
\end{aligned}
$$

A rotation of fields is not a rotation at the level of creation and annihilation operators.

Flavor Vacuum

The flavor vacuum is defined by[4]:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\; |0\rangle_{1,2}$$

In the infinite volume limit:

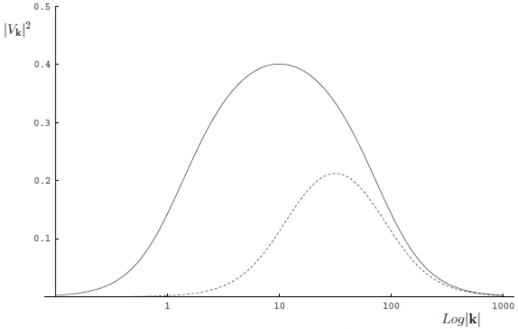
$$\lim_{V\to\infty} \;{}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\;\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

where

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

[4] M.Blasone, G.Vitiello (1995).

Vacuum condensate



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- Condensation density: ${}_{e\,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha_{\mathbf{k},i}|0\rangle_{e\,\mu} = \sin^2\theta\,|V_{\mathbf{k}}|^2$, with $i = 1, 2$. Same result for antiparticles.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

Flavor eigenstates

A neutrino flavor-state can be defined:

$$|\nu^r_{\mathbf{k},\sigma}(t)\rangle = \alpha^{r\dagger}_{\sigma,\mathbf{k}}|0(t)\rangle_{e\,\mu}\,, \qquad \sigma = e, \mu$$

At any time these are eigenstates of the flavor charges:

$$Q_\sigma(t)|\nu^r_{\mathbf{k},\sigma}(t)\rangle = |\nu^r_{\mathbf{k},\sigma}(t)\rangle$$

Note that $|\nu^r_{\mathbf{k},\sigma}(t)\rangle \neq |\nu^r_{\mathbf{k},\sigma}\rangle_P$ because

$$\left[B(m_1, m_2)\,,\; R^{-1}(\theta)\right] \;\neq\; 0$$

Bogoliubov and Pontecorvo do not commute!

[ , ] ≠ 0

As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \left[B(m_1, m_2)\,,\ R^{-1}(\theta)\right]\ |\widetilde{0}\rangle_{1,2}$$

Flavor Vacuum and Condensate Structure

The flavor vacuum is characterized by a condensate structure:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \prod_{r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2 \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]|0\rangle_{1,2}$$

- SU(2) (Perelomov) coherent state.
- Condensate structure on vacuum as in systems with SSB (e.g. superfluids, superconductors).
- Exotic condensates: mixed pairs due to a non-diagonal Bogoliubov transformation.
- Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

Why is vacuum structure so important?

- Dark Energy contribution of the neutrino mixing[5]:

$$\Lambda \;=\; 128\pi^3 G \int_0^K \mathrm{d}k\, k^2 (\omega_{k,1} + \omega_{k,2}) |V_{\mathbf{k}}|^2$$

If the cut-off is chosen of the order $K \sim \sqrt{m_1 m_2}$, with $m_1 = 7 \times 10^{-3} eV$ and $m_2 = 5 \times 10^{-2} eV$ one gets:

$$\Lambda \sim 10^{-56} cm^{-2}$$

which is compatible with modern experimental upper bounds.

[5] M.Blasone, A.Capolupo,S.Capozziello, S.Carloni, G.Vitiello, (2005)

Patterns of Dynamical Symmetry Breaking

Chiral symmetry

Consider the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + U(\overline{\psi}\psi)$$

where

$$\psi = \begin{bmatrix} \psi_{\mathrm{I}} \\ \psi_{\mathrm{II}} \end{bmatrix} .$$

$U\left(\overline{\psi}\psi\right)$ is assumed to be invariant under chiral transformations $U(2)_V \times U(2)_A$:

$$g = (g, g_5) ,$$

$$g = e^{i\omega_\alpha \frac{\sigma^\alpha}{2}} , \qquad g_5 = e^{i\omega_\alpha \frac{\sigma^\alpha\gamma^5}{2}} , \quad \alpha = 0, 1, 2, 3$$

so the entire Lagrangian is invariant as well.

Chiral Symmetry Breaking

The vector and axial Noether charges are:

$$J_\mu^\alpha = \overline{\psi}\gamma_\mu \frac{\sigma^\alpha}{2}\psi$$

$$J_{5\mu}^\alpha = \overline{\psi}\gamma_\mu\gamma_5 \frac{\sigma^\alpha}{2}\psi$$

If we add a diagonal mass term

$$\mathcal{L}_M = -m\overline{\psi}\psi$$

the conservation law of axial currents is explicitly broken:

$$\partial^\mu J_\mu^\alpha = 0$$

$$\partial^\mu J_{5\mu}^\alpha = i\overline{\psi}\,\gamma_5 m\,\psi$$

Isospin Symmetry Breaking

Adding a mass-shift term

$$\mathcal{L}_{\Delta m} \;=\; -\overline{\psi}\begin{bmatrix}\Delta m & 0 \\ 0 & -\Delta m\end{bmatrix}\psi$$

the Isospin symmetry is broken to $U(1)^0_V \times U(1)^3_V$ (the subscript index indicates the generator)

$$\begin{aligned}
\partial^\mu J^0_\mu &= 0 \\
\partial^\mu J^1_\mu &= \frac{i}{2}\Delta m\overline{\psi}\,\sigma_1\,\psi \\
\partial^\mu J^2_\mu &= \frac{1}{2}\overline{\psi}\,\Delta m\sigma_1\,\psi \\
\partial^\mu J^3_\mu &= 0
\end{aligned}$$

Total Flavor Charge Conservation

Finally we add to the Lagrangian, an off-diagonal term

$$\mathcal{L}_h \;=\; -\overline{\psi}\begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix}\psi$$

The current evolution are now

$$\begin{aligned}
\partial^\mu J^0_\mu &= 0 \\
\partial^\mu J^1_\mu &= \frac{i}{2}(m_{\rm I} - m_{\rm II})\overline{\psi}\,\sigma_1\,\psi \\
\partial^\mu J^2_\mu &= -\frac{1}{2}\overline{\psi}\,\left[2h\sigma_3 - (m_{\rm I} - m_{\rm II})\sigma_1\right]\,\psi \\
\partial^\mu J^3_\mu &= -ih\overline{\psi}\,\sigma_1\,\psi
\end{aligned}$$

Conservation of the total flavor charge Q^0.

Dynamical Generation of Fermion Mixing

Dynamical generation of mixing occurs if[6]

$$U(2)_V \times U(2)_A \longrightarrow U(1)^0_V\,,$$

at the ground state level. SSB is characterized by the existence of some (quasi)-local operators Φ_i so that

$$\langle\Omega|\,[Q^\alpha(0), \Phi_i(0)]\,|\Omega\rangle \;=\; \langle\Omega|\varphi_i^\alpha|\Omega\rangle \;\neq\; 0\,,$$

on some *dressed* vacuum. φ_i^α are called *order parameters*. We look at order parameters of the form $\overline{\psi}_i\psi_j \pm \overline{\psi}_k\psi_l$ with $i,j,k,l = \mathrm{I},\mathrm{II}$.

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[6] M. Blasone, P. Jizba, L. S., in preparation (2017).

Patterns of SSB

Symmetry Group	Order Parameter	Broken Charges
$U(2)_V \times U(2)_A$	-	-
$U(2)_V$	$\langle \overline{\psi}_{\mathrm{I}} \psi_{\mathrm{I}} + \overline{\psi}_{\mathrm{II}} \psi_{\mathrm{II}} \rangle \neq 0$	Q_5^α
$U(1)_V^0 \times U(1)_V^3$	$\langle \overline{\psi}_{\mathrm{I}} \psi_{\mathrm{I}} \pm \overline{\psi}_{\mathrm{II}} \psi_{\mathrm{II}} \rangle \neq 0$	$Q_5^\alpha;\ Q^1;\ Q^2$
$U(1)_V^0$	$\langle \overline{\psi}_{\mathrm{I}} \psi_{\mathrm{I}} \pm \overline{\psi}_{\mathrm{II}} \psi_{\mathrm{II}} \rangle \neq 0$ $\langle \overline{\psi}_{\mathrm{I}} \psi_{\mathrm{II}} + \overline{\psi}_{\mathrm{II}} \psi_{\mathrm{I}} \rangle \neq 0$	$Q_5^\alpha;\ Q^1;\ Q^2;\ Q^3$

Vacuum degeneracy (1)

All charges are time independent:

$$[Q^\alpha, H] = 0$$

$$[Q_5^\alpha, H] = 0\,, \quad \alpha = 0, 1, 2, 3$$

Degenerate vacua:

$$|\vec{\theta}, \vec{\theta}_5\rangle = e^{i\theta_\alpha Q^\alpha + i\theta_{5\,,\alpha} Q_5^\alpha} |\Omega\rangle$$

$|\Omega\rangle$ is a *fiducial* vacuum.

Vacuum degenracy (2)

Consider a fiducial vacuum $|\Omega(m)\rangle$, where only

$$v \;=\; \sum_{j=\mathrm{I,II}} \langle\Omega(m)|\overline{\psi}_j(x)\psi_j(x)|\Omega(m)\rangle \neq 0$$

We find

$$\sum_{j=\mathrm{I,II}} \langle\theta_0^5|\overline{\psi}_j(x)\gamma_5\psi_j(x)|\theta_0^5\rangle \;=\; i\sin\theta_0^5\; v$$

$$\langle\theta_3^5|\overline{\psi}_{_\mathrm{II}}(x)\gamma_5\psi_{_\mathrm{II}}(x) \;-\; \overline{\psi}_{_\mathrm{I}}(x)\gamma_5\psi_{_\mathrm{I}}(x)|\theta_3^5\rangle \;=\; i\sin\theta_3^5\; v$$

$$\langle\theta_1^5|\overline{\psi}_{_\mathrm{I}}(x)\gamma_5\psi_{_\mathrm{II}}(x) \;+\; \overline{\psi}_{_\mathrm{II}}(x)\gamma_5\psi_{_\mathrm{I}}(x)|\theta_1^5\rangle \;=\; i\sin\theta_1^5\; v$$

$$\langle\theta_2^5|\overline{\psi}_{_\mathrm{I}}(x)\gamma_5\psi_{_\mathrm{II}}(x) \;-\; \overline{\psi}_{_\mathrm{II}}(x)\gamma_5\psi_{_\mathrm{I}}(x)|\theta_2^5\rangle \;=\; \sin\theta_2^5\; v$$

in contrast with our hypothesis.

Vacuum degenracy (3)

Consider $|\Omega(m_1, m_2)\rangle$, leaved invariant Q^0, Q^3. We introduce

$$Q_+ = Q^1 \; + \; iQ^2 \, , \quad Q_- = Q^1 \; - \; iQ^2$$

The degenerate vacua are *Perelomov SU(2) coherent states*:

$$|\theta\rangle \;\equiv\; \exp\left[\theta \left(Q_+ \; - \; Q_- \right)\right] |\Omega(m_1, m_2)\rangle$$

One gets

$$\langle\theta|Q^3|\theta\rangle \;=\; \sin 2\theta \, \langle\Omega(m_1, m_2)|Q^1|\Omega(m_1, m_2)\rangle \;\neq\; 0$$

The residual symmetry, for every $\theta \neq 0$, is $U(1)^0_V$: Dynamical origin of mixing.

Two Flavor NJL model

Two-flavor NJL model[7] is described by the Lagrangian:

$$\mathcal{L} \;=\; i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - \overline{\psi}_L M\psi_R - \overline{\psi}_R M^{\dagger}\psi_L - \frac{1}{2G}\mathrm{tr}\left(MM^{\dagger}\right)$$

M is an auxiliary boson field:

$$M_{ab} = -2G\overline{\psi}^{b}_{R}\psi^{a}_{L} \qquad\qquad M^{\dagger}_{ab} = -2G\overline{\psi}^{b}_{L}\psi^{a}_{R}. \tag{1}$$

In the mean-field approximation, M is substituted by its vev:

$$M_c \;=\; \begin{bmatrix} m_e & m_{e\,\mu} \\ m_{e\,\mu} & m_{\mu} \end{bmatrix}. \tag{2}$$

[7] Y.Nambu, G.Jona Lasinio (1961).

Gap equations and dynamical generation of mixing

We derive, thanks to 1-loop Effective Action[8]

$$
\begin{aligned}
m_e &= \frac{i\,G}{4\pi^4}\int \mathrm{d}^4 p\, \frac{m_e\, p^2 - m_\mu\, d}{p^4 - p^2 t + d^2} \\
m_\mu &= \frac{i\,G}{4\pi^4}\int \mathrm{d}^4 p\, \frac{m_\mu\, p^2 - m_e\, d}{p^4 - p^2 t + d^2} \\
m_{e\,\mu} &= \frac{i\,G}{4\pi^4}\int \mathrm{d}^4 p\, \frac{p^2 + d}{p^4 - p^2 t + d^2}
\end{aligned}
$$

where

$$
d = \det M_c\,, \qquad t = \mathrm{tr} M_c^2
$$

These equations present all solutions previously discussed.

[8] M.Blasone, P.Jizba, L.S. (2015).

Conclusions and Perspectives

Conclusions

Conclusions:

- The non-trivial condensate structure of the flavor vacuum suggests a dynamical origin of mixing. The same structure has been proved to be a mark of this phenomenon.
- For models with chiral $U(2)_V \times U(2)_A$, we have shown how to derive general information on various patterns of symmetry breaking and about dynamical generation of mixing. These are confirmed in specific examples.

Perspectives

Perspectives:

- The application of FI methods has suggested the study of appearance of inequivalent representations in this framework[9].
- More general situations, eventually including Lorentz-Poincaré symmetry breaking were only slightly touched[10]. The use of FI's in studying these situations requires a generalization of the QM and QFT partition function[11].

[9] M. Blasone, P. Jizba, L. S. (2017).

[10] M.Blasone, P. Jizba, G. Lambiase, N. Mavromatos (2014).

[11] M. Blasone, P. Jizba, L. S. (2017).

Thank you for the attention!