From the Coleman-Mandula Theorem to Supersymmetric Yang-Mills Theories

by

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Lecture Notes

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Contents

Introduction

The goal of these lectures is to introduce the most important ingredients to be able to understand the work of Seiberg and Witten on the N=2 supersymmetric Yang-Mills theory. The main ingredient we shall focus on is supersymmetry (Susy) itself, especially Susy gauge theories. Nonetheless, along the way, we shall present other important ideas born in the theoretical investigation of high energy physics, such as electro-magnetic (em) duality, non-perturbative effects, topological features of field theories.
The lectures are two hours each. There are two (easy) exercises for each lecture, and suggested further reading in the measure of at least one paper for each lecture. The expected home-work is to solve all the exercises and present a paper (either one of the suggested papers or one of your choice) in the form of a journal club during the last lecture.

Chapter 1

Some of the Key Ideas in the Theory of High Energy Physics

We want to introduce some of the most fascinating ideas currently used in the theoretical investigation of high energy physics, as well as in other areas of theoretical physics. We intend to do so because many of these ideas become tools to solve the Seiberg-Witten model of N=2 Susy Yang-Mills quantum theory (in four dimensions). In particular we shall introduce, in the most (to our taste) pedagogical manner, em duality, and topological objects in field theory. Of course, these topics are of strong interest on their own...

1.1 EM Duality (Dirac Monopoles)

Consider Maxwell electrodynamics

$$-\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{1.1}$$

with the field strength and its dual defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \tag{1.2}$$

respectively. Maxwell equations in the vacuum can be written as

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu F^{*\mu\nu} = 0\,. \tag{1.3}$$

It is crucial to notice that the first set of equations is *dynamical*, while the second set is *geometrical* (Bianchi identities). These equations transform into one another under a peculiar transformation

$$i) \quad F^{\mu\nu} \to F^{*\mu\nu}\,,$$

which in components becomes

$$\vec{E} \to \vec{B} \quad \text{and} \quad \vec{B} \to -\vec{E}\,, \tag{1.4}$$

hence

$$ii) \quad F^{*\mu\nu} \to -F^{\mu\nu}\,.$$

Note that under the transformations (1.4) the theory is not invariant

$$\mathcal{L} = \frac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2) \to -\mathcal{L} \tag{1.5}$$

while the equations of motion are.
First lesson: *em duality is not a symmetry in the Noether sense.*
In the previous example em duality is rather a discrete symmetry of the equations of motion, but more generally it is not even that.
Suppose now that an electric source is present, $j_e^\mu \neq 0$, where with q_e we indicate the electric charge. The equations are no longer symmetric under em duality

$$\partial_\mu F^{\mu\nu} = j_e^\nu \quad \text{and} \quad \partial_\mu F^{*\mu\nu} = 0\,, \tag{1.6}$$

unless one (Dirac) introduces q_m, the *magnetic* charge, to modify the last equation into

$$\partial_\mu F^{*\mu\nu} = j_m^\nu\,. \tag{1.7}$$

Thus, besides the transformations (1.4) one has to add

$$q_e \to q_m \quad \text{and} \quad q_m \to -q_e\,. \tag{1.8}$$

Everything seems so nice that one wonders why magnetic monopoles are not easily spotted everywhere! Things, though, are not so simple and the subtle arguments of topology come in.
The Bianchi identities cannot be modified without paying a price. As a matter of fact

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \Leftarrow \partial_\mu F^{*\mu\nu} = 0\,, \tag{1.9}$$

e.g. $\vec{B} = \vec{\nabla} \times \vec{A}$ *if* $\vec{\nabla} \cdot \vec{B} = 0$. A singularity in space is needed (non-trivial topology).

In fact, if one computes the total magnetic flux through a sphere S^2 surrounding the origin, where the magnetic charge q_m is supposed to be sitting, then

$$\begin{aligned}\Phi_m &= \int_{S^2} F_{\mu\nu} d\sigma^{\mu\nu} = (\int_{H^-} + \int_{H^+}) F_{\mu\nu} d\sigma^{\mu\nu} \\ &= (\overleftarrow{\oint}_{\partial H} + \overrightarrow{\oint}_{\partial H}) A_\mu dx^\mu = (\int_{S^1} - \int_{S^1}) \vec{A} \cdot \vec{dl} = 0\end{aligned}$$

where the two semi-spheres $H^\pm$ have common boundary ∂H with opposite orientation.
The way out to have a non-zero magnetic charge is then to consider two solutions for the gauge potential in the two regions $H^\pm$, related by a gauge transformation, namely, using spherical coordinates

$$A_r^+ = A_\theta^+ = 0 \quad , \quad A_\phi^+ = \frac{q_m}{r} \frac{1 - \cos\theta}{\sin\theta} \, , \tag{1.10}$$

in the region H^+, and

$$A_r^- = A_\theta^- = 0 \quad , \quad A_\phi^- = \frac{q_m}{r} \frac{1 + \cos\theta}{\sin\theta} \, , \tag{1.11}$$

in the region H^-. This way one gets read of the inessential string of singularities ($r = \mp z$ in $H^\pm$, resp.), known as *Dirac string* (nothing to do with the strings of unification of the fundamental interactions). The origin of these singularities is due to the fact that it does not exist a local parameterization on the sphere S^2 which is also globally valid, one needs at least *two* of such parameterizations.
The two gauge potentials have to be related by a gauge transformation on the overlapping region, topologically equivalent to a circle (the equator S^1):

$$A_\phi^+ = A_\phi^- + i \frac{\hbar c}{q_e} T \frac{d}{d\phi} T^{-1}$$

where we (cleverly) choose the gauge group element to be

$$T = \exp\{2i \frac{q_m q_e}{\hbar c} \phi\} \, . \tag{1.12}$$

If we now compute the flux we find the wanted result

$$\Phi_m = \int_{S^1} (\vec{A}^+ - \vec{A}^-) \cdot \vec{dl} = i \frac{\hbar c}{q_e} \int_{S^1} \frac{d}{d\phi} (\ln T^{-1}) d\phi = 4\pi q_m \, . \tag{1.13}$$

Second lesson: *Magnetic charges are of topological nature.*

Third lesson: *E-M duality is an inversion of the coupling.*
This last point is easily seen by requiring the gauge group element to be single valued:

$$T(\phi) \equiv T(\phi + 2\pi m) \Rightarrow \frac{4\pi q_m q_e}{\hbar c} = 2\pi n\,, \tag{1.14}$$

where $m, n \in \mathbf{Z}$, hence

$$q_m = (n\frac{\hbar c}{2})\frac{1}{q_e}\,. \tag{1.15}$$

E-M duality is actually long known in physics, it exchanges weak and strong coupling regimes: $G \to 1/G$, where G indicates a generic coupling of the theory. In string theory and some supersymmetric gauge theories, this is referred to as S duality. We shall see later in this course that in Seiberg-Witten theory this duality is represented by only one of the generators of the whole duality group $SL(2,\mathbf{Z})$, another one corresponding to the so-called T duality. Well known examples are certain two-dimensional theories, where duality may exchange different phases of the same theory, as for the Ising model, or map solutions of a theory into solutions of a different theory, as for the bosonic Sine-Gordon and fermionic Thirring models. In the latter case duality exchanges the solitonic solutions of the Sine-Gordon model with the elementary particles of the Thirring model. Note again that, on general grounds, these transformations are not symmetries in the Noether sense, but rather ways to connect different "phases".

1.2 More on Topology ('t Hooft-Polyakov Monopoles)

A Yang-Mills gauge theory in the spontaneously broken phase admits 't Hooft-Polyakov monopole solutions. For instance, let us consider an SU(2) gauge theory with Higgs potential $V(\phi^a)$. In the spontaneously broken phase the scalar fields tend to their vacuum value $\phi^a \sim a\frac{r^a}{r}$ where $a \in \mathbf{C}$, as $r \to \infty$. On the other hand, in the same limit, the vector potentials $A^{ai} \sim \epsilon^{iab}\frac{r^b}{r^2}$, $A^{a0} = 0$. This behavior gives rise to a magnetic charge. By performing a SU(2) gauge transformation on this radially symmetric ("hedgehog") solution we can align $< 0|\phi^a|0 >$ along one direction (the Coulomb branch), say $< 0|\phi^a|0 >= \delta^{a3}a$, and the 't Hooft-Polyakov monopole becomes a U(1) Dirac-type monopole.

A "topologically correct" way to describe this equivalence is to say that the 't Hooft-Polyakov magnetic charge is the winding number of the map $SU(2)/U(1) \sim S^2 \to S^2_\infty$, that identifies the homotopy class of the map. By considering the maps $U(1) \sim S^1 \to S^1_\infty$, where S^1_∞ is the equator of S^2_∞, it is clear that a similar comment holds for the U(1) Dirac type magnetic charge. It turns out that the two homotopy groups, $\pi_2(S^2)$ and $\pi_1(S^1)$, are isomorphic to **Z**.

1.3 Exercises

Exercise I.a We have seen that em duality, in general, is not a symmetry in the Noether sense. Discuss at least another example of this kind.
Exercise I.b Give at least one qualitative argument about the topological nature of the Dirac monopole.

1.4 Further Reading

[1] L.H. Ryder, Quantum Field Theory, Cambridge University Press, 1985 (Chp. 10).

Chapter 2

From the No-go Theorems to the Birth of Supersymmetry

2.1 No-go Theorems

The ”no-go theorems” prove the impossibility of non-trivially combining Lorentz invariance and internal symmetry for physical theories. The most powerful is the Coleman-Mandula (CM) theorem [1]. The generalization of this theorem lead to the discovery of supersymmetry.
The statement of the CM theorem is as follows: *“Let $\mathcal{E}$ be a connected symmetry group of the S matrix, and let the following five conditions hold: (I) $\mathcal{E}$ contains a subgroup locally isomorphic to the inhomogeneous Lorentz (Poincaré) group $\mathcal{L}$; (II) all particle types correspond to positive-energy representations of $\mathcal{L}$, and, for any finite mass M, there are only a finite number of particle types with mass less than M; (III) elastic-scattering amplitudes are analytic functions of the center of mass energy and of the momentum transfer in some neighborhood of the physical region; (IV) at almost all energies, any two plane waves scatter; (V) the generators of $\mathcal{E}$ are representable as integral operators in momentum space, with kernels that are distributions. Then $\mathcal{E}$ is locally isomorphic to $\mathcal{L} \times \mathcal{T}$, the direct product of the Poincaré group and the internal symmetry group”.* We shall later prove and discuss this theorem in some detail.
In the 1960’s there were two kinds of motivations for investigating this problem in particle physics.
The first concerned the mass splitting occurring within the multiplets of particles. The challenge was to find a group containing the internal symmetry group, with non-trivial commutations among the generators of the latter and those of space-time translations, P_μ. For example let us consider the

isospin group, whose SU(2) algebra we write in the Cartan-Weyl basis

$$[T_+, T_-] = 2T_0 \quad \text{and} \quad [T_0, T_\pm] = \pm T_\pm \,, \tag{2.1}$$

where T_+ (T_-)is the step-up (step-down) operator, and T_0 is the generator of the Cartan sub-algebra. The commutator $[P_\mu, T_+]$ is zero for ordinary isospin symmetry. However, if it is different from zero *within* the bigger group, this would give account for the mass splitting as a higher symmetry effect. Consider the doublet of nucleons $|p\rangle \equiv |+\rangle$ and $|n\rangle \equiv |-\rangle$ as the two states of the fundamental irrep of SU(2)

$$T_0|\pm\rangle = \pm|\pm\rangle \,, \quad T_\pm|\pm\rangle = 0 \,, \quad T_\mp|\pm\rangle = |\mp\rangle \,. \tag{2.2}$$

By our hypothesis $[P_\mu, T_+] = 0 \Rightarrow P^2|\pm\rangle = m^2|\pm\rangle$, which means $m_p^2 = m_n^2$. Hence the experimental results $m_p \sim 938.3$MeV and $m_n \sim 939.6$MeV, cannot be explained. Suppose, instead, that

$$[P_\mu, T_+] = c_\mu T_+ \Rightarrow [P^2, T_+] = 2c_\mu P^\mu T_+ - c^2 T_+ \,, \tag{2.3}$$

hence

$$P^2|+\rangle = P^2 T_+|-\rangle = T_+ P^2|-\rangle + 2c_\mu P^\mu|+\rangle - c^2|+\rangle \,, \tag{2.4}$$

and by using the translation invariance of the state ($P^\mu|+\rangle \sim \partial^\mu|+\rangle = 0$) one has the splitting

$$m_p^2 = m_n^2 - \Delta m^2 \,, \tag{2.5}$$

where $\Delta m^2 \equiv c^2$. Nowadays the accepted explanation of the mass-splitting phenomenon is the breakdown of internal symmetry.

The second motivation to investigate non trivial combinations of space-time and internal symmetries was the discovery of a model where the internal 3-flavor symmetry group $SU(3)$ and the *non-relativistic* spin group $SU(2)$ were non-trivially combined into $SU(6)$, which contains but is not isomorphic to $SU(3) \times SU(2)$. This gives the so-called static quark model. As a matter of fact the barion octet $J^P = \frac{1}{2}^+$

$$\begin{array}{ccccc} & n & & p & \\ \Sigma^- & & \Sigma^0/\Lambda & & \Sigma^+ \\ & \Xi^- & & \Xi^0 & \end{array}$$

and decuplet $J^P = \frac{3}{2}^+$

$$\begin{array}{ccccccc} \Delta^- & & \Delta^0 & & \Delta^+ & & \Delta^{++} \\ & \Sigma^{*-} & & \Sigma^{*0} & & \Sigma^{*+} & \\ & & \Xi^{*-} & & \Xi^0 & & \\ & & & \Omega^- & & & \end{array} \,,$$

which differ by spin, both fit into a 56-plet of SU(6). This is easily seen if one considers that the dimension of the representation has to be $d \times (2J+1)$, where d is the dimension of the SU(3) representation, and J the non-relativistic spin. In this case: i) $d = 10$ and $2J + 1 = 4$ gives 40, while ii) $d = 8$ and $2J + 1 = 2$ gives 16, which add up to 56. On the other hand the tensorial representation of SU(6): $6 \times 6 \times 6 = 56 + 70 + 70 + 20$. Similar considerations hold for the meson nonets of $J^P = 0^-$ and $J^P = 1^-$. The natural task then was to extend this result to a fully relativistic theory.
These programs were brought to a negative end first by the O'Raifeartaigh (LOR) theorem [2] which completes and generalizes the results of the work started with the first "no-go" theorem of McGlinn (McG) [3], and later by the CM theorem stated above. All these theorems hold if one considers Lie groups as symmetry groups of the theory (LOR theorem holds for finite order Lie algebras, while CM theorem holds also for the infinite case) and are of local nature. Nevertheless if one relaxes the assumption of having only standard Lie groups, for instance by allowing for *graded* structures, then the negative-type conclusions no longer hold [4]. The most surprising feature of these graded algebras is the occurrence of transformations among particles differing by spin: this is the birth of supersymmetry. In Ref. [5] the most general supersymmetric algebra of the S matrix was introduced and its representations extensively studied, closing the era of the "no-go" theorems with a "let's go" theorem: the Haag-Lopuszánski-Sohnius (HLS) theorem. (Note that the title of the paper in Ref. [5] is "All Possible Generators of Supersymmetry of the S Matrix" as opposed to the title of the paper in Ref. [1] "All Possible Symmetries of the S Matrix"). In the following we shall state McG theorem, state and discuss LOR theorem, and finally prove and discuss CM theorem.

2.2 McGlinn and O'Raifeartaigh Theorems

McG theorem. *"Let L be the Lie algebra of the inhomogeneous Lorentz group, M and P the homogeneous and translation parts of L, respectively, and T any semisimple internal symmetry algebra. If E is a Lie algebra whose basis consists of the basis of L and the basis of T, and if $[T, M] = 0$ (i.e. the internal symmetry is Lorentz invariant) then $[T, P] = 0$. Hence $E = L \oplus T$".*
LOR theorem. *"Let L be the Lie algebra of the inhomogeneous Lorentz group, consisting of the homogeneous part M and the translation part P. Let E be any Lie algebra of finite order, with radical R and Levi factor G. If L is a subalgebra of E, then only the following four cases occur: (i) $R = P$;*

(ii) R Abelian but larger than, and containing P; (iii) R solvable but not Abelian, and containing P; (iv) $R \cap P = 0$*. In all cases,* $M \cap R = 0$*".* The main algebraic tools used in this theorem are the Levi decomposition and the freedom of redefining the generators (the redefinitions have trivial physical consequences). Levi's radical-splitting theorem in Lie algebra theory states that any Lie algebra E of finite order is the semidirect sum of a semisimple algebra G (the Levi factor) and the radical R (an invariant solvable algebra, where solvable means that for some integer k the k-derived algebra is zero). LOR theorem enables one to classify the ways in which L can be embedded in E. Case (i) is the only physical case and, up to a redefinition, reduces to $E = L \oplus T$, where T is a semisimple algebra (internal symmetry). Case (ii) cannot be reduced to the previous direct sum but is unphysical since it introduces a translation algebra of more than four dimensions. Case (iii) is the most unphysical since, for non-trivial representations, hermitian conjugation cannot be defined. Case (iv) amounts, up to a redefinition, to embedding L as a subalgebra in a simple Lie algebra. It is again unphysical due to the fact that the parameters corresponding to the P_μ have a non-compact range and this lead to serious difficulties in defining multiplets, even in the absence of mass-splitting. Thus, while it is possible to embed L in a larger algebra E, the ways in which this may be done are restricted and only the direct sum has a clear physical meaning. The McG result can be obtained as a special case of LOR theorem by using the first McG assumption alone and the redefinition freedom.
If one now considers the Hilbert space $\mathcal{H}$ on which any irreducible representation of the group generated by E operates, and if the mass operator P^2 has a discrete eigenvalue m^2 and is self-adjoint on $\mathcal{H}$, then the eigenspace $\mathcal{H}_m$ belonging to m^2 is closed and is invariant with respect to the elements representing E on $\mathcal{H}$. Hence the elements representing E cannot produce the mass-splitting. Sometimes in literature this result (the mass-splitting theorem) is referred to as the LOR theorem.

2.3 Proof of the Coleman-Mandula Theorem

This theorem is the most powerful of the "no-go" theorems for two reasons: firstly it deals with infinite-parameter Lie groups; secondly it uses information about the S matrix, thus it takes into account the n-particle spectrum rather than the 1-particle spectrum of LOR theorem, and does not depends on the details of the underneath quantum field theory. The Hilbert space of scattering theory is an infinite direct sum of n-particle subspaces $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n)}$, where $\mathcal{H}^{(n)}$ is the (symmetrized) tensor product of

n 1-particle subspaces $\mathcal{H}^{(1)}$ (up to isomorphisms). The S matrix is a unitary operator on $\mathcal{H}$, defined in the usual way

$$S = 1 - i(2\pi)^4 \delta^{(4)}(P_\mu - P'_\mu) T \ , \tag{2.6}$$

where the matrix elements of T are the scattering amplitudes and the $\delta^{(4)}$ ensures four-momentum conservation (elastic scattering). A unitary operator U on $\mathcal{H}$ is a symmetry of S if: (i) U maps 1-particle states into 1-particle states; (ii) U acts on many-particle states as if they were tensor products of 1-particle states; (iii) U commutes with S. If S is Lorentz invariant, i.e. if it has a symmetry group locally isomorphic to $\mathcal{L}$, then the plane waves $|\alpha, \lambda, p>$ are a basis of $\mathcal{H}^{(1)}$, where p is the four momentum, λ is the spin index, and α labels the irreducible subspaces of $\mathcal{H}^{(1)}$ under the action of $\mathcal{L}$ and counts the number of particle types in $\mathcal{H}^{(1)}$. An internal symmetry is defined as a symmetry of S that commutes with $\mathcal{L}$, and thus it acts on the α indices and has vanishing matrix elements between particles of different ps or λs. Note here that this definition of internal symmetry seems to immediately give the result one will eventually prove! This is not so, because one has to investigate what happens *within* the higher group $\mathcal{E}$ having $\mathcal{T}$ and $\mathcal{L}$ as subgroups. Finally let us define $\mathcal{D}$ as the subset of $\mathcal{H}^{(1)}$ consisting of the 1-particle states whose momentum-space wave-functions are infinitely differentiable with compact support (test functions).
To prove the theorem one considers at first the 2-particle scattering. By using assumption (V) one sees that for two 2-particle states $x_1 \otimes x_2$ and $y_1 \otimes y_2$ in $\mathcal{D} \otimes \mathcal{D}$,

$$(y_1 \otimes y_2, A[x_1 \otimes x_2]) = (y_1, Ax_1)(y_2, x_2) + (y_1, x_1)(y_2, Ax_2) \ , \tag{2.7}$$

where the distribution A represents the infinitesimal generator of $\mathcal{E}$ in some neighborhood of the identity. In this formalism the fact that $\mathcal{E}$ is a symmetry of the S matrix is equivalent to

$$(S[y_1 \otimes y_2], AS[x_1 \otimes x_2]) = (y_1 \otimes y_2, A[x_1 \otimes x_2]) \ . \tag{2.8}$$

Assumption (I) on the Lorentz invariance of S, implies that if A obeys the two conditions given above, so does $U(\Lambda, a)^\dagger A U(\Lambda, a)$, where, as usual, $U(\Lambda, a)$ is the unitary representation of the element (Λ, a) of $\mathcal{L}$. Let the set of all those distributions be denoted by $\mathcal{A}$. For a given test function $f(p)$ with support in D, not containing the origin, the distribution $f \cdot A = \int d^4 a U(1, a)^\dagger A U(1, a) \tilde{f}(a)$, where $\tilde{f}(a)$ is the Fourier transformed of $f(p)$, has matrix elements in the plane-wave basis given by

$$f \cdot A(p, p') = f(p - p') A(p, p') \ , \tag{2.9}$$

because $U(1,a)|\alpha,\lambda,p >= e^{-ip\cdot a}|\alpha,\lambda,p >$. Hence $f\cdot A$ only acts between states whose supports are connected by a vector $p-p'$ in D. By assumption (II) on particle-finiteness, if D is sufficiently small there are regions on a given mass hyperboloid such that for p in these regions and k in D, $p+k$ is not in any mass hyperboloid. This state must be annihilated by $f\cdot A$. Taking p in the complement of these regions and q, p', q' *in* these regions such that $p+q = p'+q'$, and defining the center of mass energy $s = (p+q)^2$ and the momentum transfer $t = (p-p')^2$, one has that if s is below the threshold for pair production, for these values of s and t the T matrix element must be zero. By assumption (III) on analyticity, the S matrix is trivial in the whole region of analyticity, in contradiction with assumption (IV) on the occurrence of scattering. By induction one extends this result to the many particle scattering. Hence one concludes that (*Lemma 1*) the support of $A(p,p')$ is restricted to $p = p'$. We have then re-obtained the mass-splitting theorem, since Lemma 1 implies that $A(p)$ cannot connect states on different mass hyperboloids. Since $A(p)$ is a distribution whose support is a point it is possible to write it as a polynomial in the tangential differential operator $\nabla_\mu = \partial/\partial p^\mu - m^{-2}p_\mu p_\nu \partial/\partial p_\nu$ $(p^\mu \nabla_\mu = 0)$

$$A(p) = \sum_{n=0}^{N} A'^{(n)}(p)_{\mu_1\cdots\mu_n} \nabla^{\mu_1}\cdots\nabla^{\mu_n} = \sum_{n=0}^{N} A^{(n)}(p)_{\mu_1\cdots\mu_n} \frac{\partial}{\partial p_{\mu_1}}\cdots\frac{\partial}{\partial p_{\mu_n}} \,, \tag{2.10}$$

where the $A^{(n)}$s are matrix-valued distributions and $[A(p), p_\mu p^\mu] = 0$ on $\mathcal{D}$. Now let $\mathcal{B}$ be the subset of transformations in $\mathcal{A}$ which commute with the space-time translations, and $B(p)$ its elements. It is possible to show that every $B(p)$ is the sum of an infinitesimal translation and an infinitesimal internal symmetry [1]. This is achieved by first considering $\mathcal{B}'$, the subset of $\mathcal{B}$ consisting of infinitely differentiable functions, and by splitting the $B(p)$s in $\mathcal{B}'$ into the sum of a multiple of the identity and a traceless part: $\alpha \mathrm{Tr} B(p) I + B^*(p)$ (α is a representation-dependent normalization constant). $B^*(p)$ turns out to be an infinitesimal internal symmetry, while $\mathrm{Tr}B(p)$ turns out to be a linear function of p. Finally, this result is valid also in the whole $\mathcal{B}$. Thus (*Lemma 2*), in $\mathcal{B}$ one has $B(p) = a_\mu p^\mu + b$, where a_μ is a constant four-vector and b is a constant hermitian matrix that does not involve spin indices. $A^{(N)}(p)_{\mu_1\cdots\mu_N} = [p_{\mu_1}, \cdots, [p_{\mu_N}, A(p)]\cdots]$. commutes with the space-time translations, hence it is an element of $\mathcal{B}$ and by Lemma 2 can be written as

$$A^{(N)}(p)_{\mu_1\cdots\mu_N} = a_{\lambda\mu_1\cdots\mu_N} p^\lambda + b_{\mu_1\cdots\mu_N} \,. \tag{2.11}$$

By using the symmetry properties of N-fold commutators with p^μ one obtains that $a_{\lambda\mu_1\cdots\mu_N} \neq 0$ iff $N = 0,1$ and $b_{\mu_1\cdots\mu_N} \neq 0$ iff $N = 0$. Thus (*Lemma*

3) any $A(p)$ in $\mathcal{A}$ is the direct sum of an infinitesimal Lorentz transformation, an infinitesimal translation, and an infinitesimal internal symmetry transformation. Lemma 3 is the infinitesimal version of the theorem. Q.E.D. Note that even if the theorem is stated for Lie groups rather than Lie algebras, its conclusions are of local nature, and no longer hold if local isomorphism is replaced by global isomorphism. Note also that for Galilean-invariant spin-independent theories the theorem does not hold. Finally, the assumption (IV) on non trivial scattering is $T|p, p' >\neq 0$, which is much stronger than $S \neq 1$.

2.4 Exercises

Exercise II.a Explain why for non-relativistic (Galilean invariant) theories the CM theorem does not hold. (Hint. This has to do with the compactness of the space-time group.)
Exercise II.b The Gervais-Sakita action for the string is

$$\int d\sigma^+ d\sigma^- \left(\alpha \partial_{\sigma^+} X^\mu \partial_{\sigma^-} X_\mu + i\psi_1^\mu \partial_{\sigma^-} \psi_{1\mu} + i\psi_2^\mu \partial_{\sigma^+} \psi_{2\mu}\right) , \tag{2.12}$$

where $\mu = 0, ..., d-1$, $\sigma^\pm \equiv \tau \pm \sigma$ are the two-dimensional light-cone coordinates on the string world-sheet, α is the string tension, $X^\mu(\sigma^+, \sigma^-)$ are the bosonic coordinates of the string, and $\psi^\mu_{1,2}(\sigma^+, \sigma^-)$ are d doublets of fermionic fields. Prove that this action is invariant under the world-sheet Susy transformations

$$\begin{aligned} \delta\psi_1^\mu(\sigma^+, \sigma^-) &= i\alpha\epsilon_1(\sigma^+)\partial_{\sigma^+} X^\mu(\sigma^+, \sigma^-) \\ \delta\psi_2^\mu(\sigma^+, \sigma^-) &= i\alpha\epsilon_2(\sigma^-)\partial_{\sigma^-} X^\mu(\sigma^+, \sigma^-) \\ \delta X^\mu(\sigma^+, \sigma^-) &= \epsilon_1(\sigma^+)\psi_1^\mu(\sigma^+, \sigma^-) + \epsilon_2(\sigma^-)\psi_2^\mu(\sigma^+, \sigma^-) \,. \end{aligned} \tag{2.13}$$

2.5 Further Reading

[1] S. Coleman and J. Mandula Phys. Rev. 159 (1967) 1251; [2] L. O'Raifeartaigh Phys. Rev. 139 (1965) 1052; [3] W. D. McGlinn Phys. Rev. Lett. 12 (1964) 467; [4] Yu. A. Gol'fand and E. P. Likhtman JETP Lett. 13 (1971) 323; [5] R. Haag, J. Lopuszánski and M. Sohnius Nucl. Phys. B 88 (1975) 257; [6] S. Weinberg, The Quantum Theory of Fields, Vol. III Supersymmetry, (Chp. 24 ≡ first Chp.)

Chapter 3

Susy

3.1 Building up the Super-Poincarè Algebra.

Let us relax one of the conditions of the CM theorem: introduce anticommuting generators as possible symmetries of the S matrix. This corresponds to consider a graded algebra with *odd* generators

$$Q, Q', Q''$$

besides the ordinary *even* ones

$$X, X', X'' ,$$

with the algebra among them being given by

$$[Q, Q']_+ = X \quad [X, X']_- = X'' \quad [Q, X']_- = Q''$$

where the commutator (anti-commutator) is denoted by $[,]_-$ ($[,]_+$). It is important to realize that the CM theorem will not be proved false, it will rather be generalized. Hence within the super-group we will indeed be able to mix non-trivially internal and space-time symmetries, but in a more subtle way than $[P_\mu, T_+] \neq 0$, and the CM results can still be used. For instance the even generators have to be

$$X \in \quad \begin{array}{l} L = P \bowtie M \\ \text{or} \\ A = A_1 \oplus A_2 \, , \end{array} \tag{3.1}$$

where L is the Poincarè algebra, semidirect product of Lorentz and translations, while the elements of the algebra of internal symmetry A are Lorentz scalars, A_1 is semi-simple, and A_2 is Abelian.
From the point of view of Lorentz properties

P_μ transforms under the $(\frac{1}{2},\frac{1}{2})$ vector irrep.

$M_{\mu\nu}$ transforms under the $(1,0)+(0,1)$ second-rank tensor irrep.

the elements of A transform under the $(0,0)$ scalar irrep.

The odd generators instead

$$Q = \sum Q_{\alpha_1\ldots\alpha_a,\dot\alpha_1\ldots\dot\alpha_b} \tag{3.2}$$

transform under the spin $\frac{1}{2}(a+b)$ irrep of Lorentz, and, of course $(a+b)$ is odd. Suppose that $\bar{Q}$ belongs to the algebra too, and consider the spin $(a+b)$ object

$$Q_{\underbrace{1\ldots1}_{a\text{ times}},\underbrace{\dot1\ldots\dot1}_{b\text{ times}}}\bar{Q}_{\underbrace{\dot1\ldots\dot1}_{a\text{ times}},\underbrace{1\ldots1}_{b\text{ times}}}$$

If this product has to belong to the algebra hence $(a+b)$ can only be either 0 or 1, because

$$[Q,\bar{Q}]_+ = X$$

and the spin $(a+b)$ X can either be a scalar (in A) or a vector under Lorentz (the second-rank tensor is ruled out by the hypothesis that Q is odd). Hence the first anti-commutation relation of this algebra is easily written as

$$[Q^L_\alpha, \bar{Q}_{\dot\alpha M}]_+ = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu\delta^L_M\,, \tag{3.3}$$

where $L,M = 1,\ldots,N$, and for $N>1$ we say that Susy is extended. Note the space-time nature of the supercharges.
In order to construct the other relations one (cleverly) uses the following super Jacobi identities

$$[A,[B,C]_\pm]_\pm \pm [B,[C,A]_\pm]_\pm \pm [C,[A,B]_\pm]_\pm = 0\,, \tag{3.4}$$

where the use of the commutator $[,]_-$ or the anti-commutator $[,]_+$ depends on the even/odd nature of the generators involved, and the signs among the terms depends on the number of commutations of the odd generators.
Let us look at the relation

$$[\underbrace{Q^L_\alpha}_{(\frac{1}{2},0)}, \underbrace{Q^M_\beta}_{(\frac{1}{2},0)}]_+ = \epsilon_{\alpha\beta}\underbrace{Z^{[LM]}}_{(0,0)} + \underbrace{M_{(\alpha\beta)}Y^{(LM)}}_{(1,0)} \tag{3.5}$$

where we indicate the properties under Lorentz. Since there are no spin $\frac{3}{2}$ generators we have

$$[\underbrace{Q^L_\alpha}_{(\frac{1}{2},0)}, \underbrace{P_\mu}_{(\frac{1}{2},\frac{1}{2})}]_- = 0 \tag{3.6}$$

and, by using the super Jacobi identities

$$0 = [P_\mu, [Q^L_\alpha, Q^M_\beta]_+]_- = [P_\mu, M_{\rho\sigma}]_- \sigma^{\rho\sigma} Y^{(LM)} \quad \Rightarrow \quad Y^{(LM)} = 0$$

since $[P_\mu, M_{\rho\sigma}]_- \neq 0$. We can now write the N-extended Susy algebra

$$[Q^L_\alpha, \bar{Q}_{\dot\alpha M}]_+ = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^L_M \tag{3.7}$$
$$[Q^L_\alpha, Q^M_\beta]_+ = \epsilon_{\alpha\beta} Z^{[LM]} \tag{3.8}$$
$$[\bar{Q}_{\dot\alpha L}, \bar{Q}_{\dot\beta M}]_+ = \epsilon_{\dot\alpha\dot\beta} Z^*_{[LM]} \tag{3.9}$$
$$[Q^L_\alpha, P_\mu]_- = [\bar{Q}_{\dot\alpha L}, P_\mu]_- = 0 \tag{3.10}$$
$$[Q^L_\alpha, B_l]_- = (S_l)^L_M Q^M_\alpha \tag{3.11}$$
$$[B^l, \bar{Q}_{\dot\alpha L}]_- = (S^{*l})^M_L \bar{Q}_{\dot\alpha M} \tag{3.12}$$
$$[B_l, B_m]_- = i C^k_{lm} B_k \tag{3.13}$$

where the B_ls are the internal symmetry generators belonging to A_1, and the Poincarè "sector" of the algebra has been omitted. Note that, if one wants to consider the largest (super) symmetry of the S-matrix, the full conformal group has to be included[1].
By using the super-Jacobi for $B_l, Q^L_\alpha, Q^M_\beta$, it is easy to prove that the $Z^{[LM]}$s close and invariant sub-algebra of $A = A_1 + A_2$. By using the super-Jacobi for $Q^L_\alpha, Q^M_\beta, \bar{Q}_{\dot\gamma K}$, one easily proves that $[\bar{Q}_{\dot\gamma K}, Z^{[LM]}]_- = 0$ and $\frac{1}{2}\epsilon^{\beta\alpha}[[Q^K_\alpha, Q^N_\beta]_+, Z^{[LM]}]_- = [Z^{[KN]}, Z^{[LM]}]_- = 0$, hence, since from the hypothesis A_1 is semi-simple, the $Z^{[LM]}$s span A_2, the Abelian sub-algebra of A, $[B_l, Z^{[LM]}]_- = 0$.
The generators $Z^{[LM]}$ are *central charges* of the Susy algebra. They play a major role in Susy gauge theories, where the Higgs mechanism to give mass can actually take place *within* Susy, if the central charges are non-zero (see next Section). It is important to note that, even when at the algebraic level it is possible to have $Z^{[LM]} \neq 0$, as in the N=2 case, is the actual physical realization of the algebra (i.e. the underlying model) that tells us if the central charges are different from zero. For instance, in the phase where the gauge symmetry is not broken, even the N=2 SYM possesses trivial central charges.
Finally, in the Exercises of this Chapter we propose to prove that the $(S_l)^L_M$s form a representation of A.

[1]This means that besides the generators of the Poincarè group, P_μ and $M_{\mu\nu}$, also the dilation generator D, and the special conformal generators C_μ have to be considered. See also the Chapter on Noether currents.

3.2 Irreducible Representations and Supermultiplets.

We have now a fully relativistic (even conformal) way to combine space-time and internal symmetries. We already saw that in the static non-relativistic quark model the SU(6) (broken) symmetry combined together particles differing by spin (for instance the spin 1/2 octet and the spin 3/2 decuplet of baryons). We have now a more powerful way of doing that within the Susy multiplets.
Firstly we notice that for any finite dimensional representation of Susy (so that the trace is well defined), we can prove that there are an equal number of bosonic and fermionic degrees of freedom. Note here that we do not need to have an equal number of bosonic and fermionic particles, but only of degrees of freedom. To prove that it is just matter of defining a fermion number operator N_F, so that

$$(-)^{N_F}|\mathrm{BOSE}>=+1|\mathrm{BOSE}> \quad \text{and} \quad (-)^{N_F}|\mathrm{FERMI}>=-1|\mathrm{FERMI}>$$

We can now compute the following trace

$$\begin{aligned} \mathrm{Tr}\{(-)^{N_F}[Q^L_\alpha,\bar{Q}_{\dot{\alpha}M}]_+\} &= \mathrm{Tr}\{(-)^{N_F}Q^L_\alpha\bar{Q}_{\dot{\alpha}M}+(-)^{N_F}\bar{Q}_{\dot{\alpha}M}Q^L_\alpha\} \\ &= \mathrm{Tr}\{-Q^L_\alpha(-)^{N_F}\bar{Q}_{\dot{\alpha}M}+Q^L_\alpha(-)^{N_F}\bar{Q}_{\dot{\alpha}M}\}=0 \end{aligned}$$

where the minus sign in the first term of the second line is due to the fact that Q_α transforms bosons $\leftrightarrow$ fermions, and the second term in the same line is due to the cyclicity of the trace. By using the first relations of the Susy algebra we then have

$$2\sigma^\mu_{\alpha\dot{\alpha}}\delta^L_M\mathrm{Tr}\{(-)^{N_F}P_\mu\}=0 \quad \Rightarrow \mathrm{Tr}\{(-)^{N_F}\}=0 \tag{3.14}$$

which proves our statement,

$$(-)^{N_F}=\begin{pmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}. \tag{3.15}$$

The irreducible representations (irreps) of extended Susy are easily found in terms of suitable linear combinations of the supercharges Q^L_α, $L=1,...,N$, to obtain creation and annihilation operators acting on a Clifford vacuum Ω_0. There are three possible cases: I) $M \neq 0$ and $Z = 0$; II) $M = 0$ and $Z = 0$; III) $M \neq 0$ and $Z \neq 0$.

I). Massive, Central-Charge-less

We move in the rest frame $P_\mu = (-M, 0, 0, 0)$, and write the Susy algebra as

$$
\begin{aligned}
[Q^L_\alpha \ , \ (Q^M_\beta)^\dagger]_+ &= 2M \ \delta^L_M \delta^\beta_\alpha && (3.16)\\
[Q^L_\alpha \ , \ Q^M_\beta]_+ &= [(Q^L_\alpha)^\dagger \ , \ (Q^M_\beta)^\dagger]_+ = 0 && (3.17)
\end{aligned}
$$

where $L, M = 1, N$. By defining $a^L_\alpha \equiv (2M)^{-1/2} Q^L_\alpha$, and $(a^L_\alpha)^\dagger \equiv (2M)^{-1/2} \bar{Q}_{\dot\alpha L}$, we have 2 (for α) $\times$ N (for L) = 2 N fermionic annihilation/creation operators, in terms of which the Susy algebra above becomes

$$
[a^L_\alpha, a^{\dagger M}_\beta]_+ = \delta^\beta_\alpha \delta^L_M \quad [a^L_\alpha, a^M_\beta]_+ = [a^{\dagger L}_\alpha, a^{\dagger M}_\beta]_+ = 0 \, . \tag{3.18}
$$

The generic state is then given by

$$
\Omega^{(n)}{}^{\alpha_1}_{A_1} \cdots {}^{\alpha_n}_{A_n} = \frac{1}{\sqrt{n!}} (a^{\alpha_1}_{A_1})^\dagger \cdots (a^{\alpha_n}_{A_n})^\dagger \Omega_0 \, , \tag{3.19}
$$

where $a^A_\alpha \Omega_0 = 0$. Of course, the multiplicity (degeneracy) of this state is $\begin{pmatrix} 2N \\ n \end{pmatrix}$. Thus the dimension of this irrep is

$$
d_I = \sum_{n=0}^{2N} \begin{pmatrix} 2N \\ n \end{pmatrix} = 2^{2N} \, . \tag{3.20}
$$

II). Mass-less, Central-Charge-less

To find the dimension of this irrep it suffices to notice that $P_\mu = (-E, 0, 0, E))$, hence

$$
[Q^L_\alpha, \bar{Q}_{\dot\alpha M}]_+ = 2 \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\alpha} \delta^L_M \, , \tag{3.21}
$$

and zero the others. This means that only one element on the right hand side is different from zero, while in the massive case there are two entries of δ^β_α different from zero. Thus the number of annihilation/creation operators is reduced to one half with respect to the massive case, and we can conclude that the dimension of this irrep is

$$
d_{II} = \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} = 2^N \, , \tag{3.22}
$$

or $d_{II} = \sqrt{d_I}$. The discrepancy of the dimension of the massive and massless irreps causes problems in the Higgs mechanism. To cure it one has to consider the next case.

III). Massive, Central-Charge
In the rest frame we can write

$$[Q^L_\alpha \, , \, (Q^M_\beta)^\dagger]_+ = 2M \, \delta^L_M \delta^\beta_\alpha \quad (3.23)$$
$$[Q^L_\alpha \, , \, Q^M_\beta]_+ = \epsilon_{\alpha\beta} \, Z^{[LM]} \quad (3.24)$$
$$[(Q^L_\alpha)^\dagger \, , \, (Q^M_\beta)^\dagger]_+ = \epsilon^{\alpha\beta} \, Z^*_{[LM]} \quad (3.25)$$

where $L, M = 1, ..., N$. Let us consider N even, and take $L \equiv (a, l)$, $M \equiv (b, m)$, where $a, b = 1, 2$, and $l, m = 1, ..., N/2$.
By performing a unitary transformation on the Q^L_α we can introduce new charges $\tilde{Q}^L_\alpha = U^L_M Q^M_\alpha$ so that in this basis $Z^{[LM]}$ is mapped to $\epsilon^{ab} 2|Z_n|$, where $Z_n = |Z_n| e^{i\zeta_n}$, $|Z_n| \geq 0$, $\zeta_n = 0$, $n = 1, ..., N/2$.
We can now define the following annihilation operators

$$a^l_\alpha = \frac{1}{\sqrt{2}} (\tilde{Q}^{1l}_\alpha + \epsilon_{\alpha\gamma} (\tilde{Q}^{2l}_\gamma)^\dagger) \quad (3.26)$$
$$b^l_\alpha = \frac{1}{\sqrt{2}} (\tilde{Q}^{1l}_\alpha - \epsilon_{\alpha\gamma} (\tilde{Q}^{2l}_\gamma)^\dagger) \quad (3.27)$$

and their conjugates $a^\dagger_\alpha$ and $b^\dagger_\alpha$, in terms of which we can write the algebra as

$$[a^l_\alpha \, , \, a^m_\beta]_+ = [b^l_\alpha \, , \, b^m_\beta]_+ = [a^l_\alpha \, , \, b^m_\beta]_+ = 0 \quad (3.28)$$
$$[a^l_\alpha \, , \, a^{m\dagger}_\beta]_+ = \delta^{lm} \delta_{\alpha\beta} 2(M + |Z_n|) \quad (3.29)$$
$$[b^l_\alpha \, , \, b^{m\dagger}_\beta]_+ = \delta^{lm} \delta_{\alpha\beta} 2(M - |Z_n|) \quad (3.30)$$

For $\alpha = \beta$ the anticommutators (3.29) and (3.30) are never less than zero on any states. Therefore from (3.29) follows $M + |Z_n| \geq 0$ and from (3.30) follows $M - |Z_n| \geq 0$. By multiplying these two inequalities together we obtain

$$M \geq |Z_n| \quad (3.31)$$

The states for which the inequality is saturated are called Bogomolnyi-Prasad-Sommerfield (BPS) states, and we have the so-called *short* Susy multiplet. The *shortness* is easily seen by considering a certain number $r \leq n$ of BPS states, for which $M = |Z_i|$, $i = 1, ..., r$. From Eq. (3.30) one immediately sees that $2r$ operators b^l_α must vanish. Thus, if $r = n = N/2$ all the $N/2$ operators b^l_α must vanish. This reduces the number of creation and annihilation operators of the Clifford algebra from $2N$ to N. Therefore the dimension of the massive representation reduces to the dimension of the massless one: from 2^{2N} to 2^N, and we can implement the Higgs mechanism without breaking Susy: $d_{III}^{\mathrm{BPS}} = d_{II}$.

3.3 Spinor Notation and Conventions.

Let us say here that in Susy conventions and notations are not a trivial matter at all. We follow the conventions of Wess and Bagger with no changes. Fortunately these conventions are becoming more and more popular and this is one of the reasons why we chose them. Rather than filling pages with well known formulae we refer to the Appendices A and B in Wess and Bagger. Here we shall comment on some of those conventions and show the formulae more relevant for our computations.

3.3.1 Crucial conventions

The spinors are Weyl two components in Van der Waerden notation. Spinors with undotted indices transform under the representation $(\frac{1}{2}, 0)$ of (the covering of) the homogeneous Lorentz group, $SL(2, \mathbf{C})$, and spinors with dotted indices transform under the conjugate representation $(0, \frac{1}{2})$. The relations between Dirac, Majorana and Weyl spinors are given by

$$\Psi_{\mathrm{Dirac}} = \begin{pmatrix} \psi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \quad \Psi_{\mathrm{Majorana}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{3.32}$$

The sigma matrices are standard Pauli matrices

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.33}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.34}$$

The relation with the gamma matrices is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{3.35}$$

The metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. To rise and lower the spinor indices we use $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$, where $\epsilon_{21} = \epsilon^{12} = -\epsilon_{12} = -\epsilon^{21} = 1$. Also $\epsilon_{0123} = -1$.

The position of the spinor indices is not negotiable and is given once and for all by

$$\sigma^{\mu}_{\alpha\dot{\alpha}} \quad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \quad \sigma^{\mu\nu\,\beta}_{\alpha} \quad \bar{\sigma}^{\mu\nu\,\dot{\alpha}}_{\dot{\beta}} \tag{3.36}$$

where

$$\sigma^{\mu\nu\,\beta}_{\alpha} = \frac{1}{4}(\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta\,\nu} - \sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta\,\mu}) \tag{3.37}$$

$$\bar{\sigma}^{\mu\nu\,\dot{\alpha}}_{\dot{\beta}} = \frac{1}{4}(\bar{\sigma}^{\dot{\alpha}\alpha\,\mu}\sigma^{\nu}_{\alpha\dot{\beta}} - \bar{\sigma}^{\dot{\alpha}\alpha\,\nu}\sigma^{\mu}_{\alpha\dot{\beta}}) \tag{3.38}$$

From σ to $\bar{\sigma}$ and *vice versa*:

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\beta} \quad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}} \tag{3.39}$$

$$\epsilon_{\alpha\beta}\bar{\sigma}^{\dot{\alpha}\beta}_{\mu} = \epsilon^{\dot{\alpha}\dot{\beta}}\sigma_{\mu\alpha\dot{\beta}} \quad \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\dot{\alpha}\beta}_{\mu} = \epsilon^{\alpha\beta}\sigma_{\mu\alpha\dot{\beta}} \tag{3.40}$$

To raise and lower spinor indices use A(9) in Wess and Bagger always matching the indices from left to right as follows:

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta} \quad \psi_{\alpha} = \epsilon_{\alpha\beta}\psi^{\beta} \tag{3.41}$$

of course

$$\psi^{\beta}\epsilon_{\beta\alpha} = \epsilon_{\beta\alpha}\psi^{\beta} = -\epsilon_{\alpha\beta}\psi^{\beta} = -\psi_{\alpha} \tag{3.42}$$

Momenta are on a different footing and the convention to raise and lower their indices is the opposite to the standard one. Namely

$$\pi^{\alpha} = \epsilon^{\beta\alpha}\pi_{\beta} \quad \pi_{\alpha} = \epsilon_{\beta\alpha}\pi^{\beta} \tag{3.43}$$

Quantities with one spinor index are Grassman variables thus anti-commute:

$$\psi_{\alpha}\chi_{\beta} = -\chi_{\beta}\psi_{\alpha}, \quad \bar{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} = -\bar{\chi}_{\dot{\beta}}\bar{\psi}_{\dot{\alpha}}, \quad \bar{\psi}_{\dot{\alpha}}\chi_{\beta} = -\chi_{\beta}\bar{\psi}_{\dot{\alpha}} \tag{3.44}$$

But some care is needed due to the (subtle) convention

$$\psi\chi \equiv \psi^{\alpha}\chi_{\alpha} = -\psi_{\alpha}\chi^{\alpha} \tag{3.45}$$

and

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \tag{3.46}$$

that leads to

$$(\psi\chi)^{\dagger} = \bar{\psi}\bar{\chi} \tag{3.47}$$

with no minus sign. Note that $\psi\chi = \chi\psi$ ($\bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}$) but $\pi\chi = -\chi\pi$ ($\bar{\pi}\bar{\chi} = -\bar{\chi}\bar{\pi}$) where π is a momentum. Explicitly writing the indices that means: $\pi^{\alpha}\chi_{\alpha} = \pi_{\alpha}\chi^{\alpha}$ and $\bar{\pi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\pi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}$.

Quantities with two spinor indices are c-number matrices

$$\epsilon_{\alpha\beta}, \quad \epsilon_{\dot\alpha\dot\beta}, \quad \sigma^{\mu}_{\alpha\dot\beta}, \quad \bar\sigma^{\mu\dot\alpha\beta}, \quad (\sigma^{\mu\nu})_{\alpha}{}^{\beta}, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}, \tag{3.48}$$

For instance the (anti)commutator of σ^μ and $\bar\sigma^\nu$ is with respect to the Minkowski indices μ, ν.
Other formulae:

$$\partial\!\!\!/_{\alpha\dot\alpha}\bar{\partial\!\!\!/}^{\dot\alpha\beta} = -\delta^{\beta}_{\alpha}\Box \quad \bar{\partial\!\!\!/}^{\dot\alpha\alpha}\ \partial\!\!\!/_{\alpha\dot\beta} = -\delta^{\dot\alpha}_{\dot\beta}\Box \tag{3.49}$$

where $\partial\!\!\!/_{\alpha\dot\alpha} \equiv \sigma^{\mu}_{\alpha\dot\alpha}\partial_\mu$ and $\bar{\partial\!\!\!/}^{\dot\alpha\alpha} \equiv \bar\sigma^{\dot\alpha\alpha}_{\mu}\partial^\mu$.
Also $\psi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\psi$, $(\psi\sigma^{\mu\nu}\chi)^\dagger = -\bar\chi\bar\sigma^{\mu\nu}\bar\psi$, $(\psi\ \partial\!\!\!/\bar\psi)^\dagger = -\bar\psi\bar{\partial\!\!\!/}\psi$.

3.3.2 Useful algebra

Beside the Fierz identities given in (B.13) in Wess and Bagger we also find

$$\psi^\alpha\lambda^\beta - \psi^\beta\lambda^\alpha = -\epsilon^{\alpha\beta}\psi\lambda \quad \psi_\alpha\lambda_\beta - \psi_\beta\lambda_\alpha = \epsilon_{\alpha\beta}\psi\lambda \tag{3.50}$$

$$\psi_\alpha\lambda^\beta - \psi^\beta\lambda_\alpha = -\delta^{\beta}_{\alpha}\psi\lambda \quad \psi^\alpha\lambda_\beta - \psi_\beta\lambda^\alpha = \delta^{\alpha}_{\beta}\psi\lambda \tag{3.51}$$

$$\bar\psi^{\dot\alpha}\bar\lambda^{\dot\beta} - \bar\psi^{\dot\beta}\bar\lambda^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi\bar\lambda \quad \bar\psi_{\dot\alpha}\bar\lambda_{\dot\beta} - \bar\psi_{\dot\beta}\bar\lambda_{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\bar\psi\bar\lambda \tag{3.52}$$

$$\bar\psi_{\dot\alpha}\bar\lambda^{\dot\beta} - \bar\psi^{\dot\beta}\bar\lambda_{\dot\alpha} = \delta^{\dot\beta}_{\dot\alpha}\bar\psi\bar\lambda \quad \bar\psi^{\dot\alpha}\bar\lambda_{\dot\beta} - \bar\psi_{\dot\beta}\bar\lambda^{\dot\alpha} = -\delta^{\dot\alpha}_{\dot\beta}\bar\psi\bar\lambda \tag{3.53}$$

Using the definitions and the properties given in (A.11), (A.14) and (A.15) in Wess and Bagger we find :

$$\sigma^{\mu\nu}\sigma^\lambda = \frac{1}{2}(-\eta^{\lambda\nu}\sigma^\mu + \eta^{\lambda\mu}\sigma^\nu + i\epsilon^{\lambda\mu\nu\kappa}\sigma_\kappa) \tag{3.54}$$

$$\sigma^{\mu}\bar\sigma^{\nu\lambda} = \frac{1}{2}(\eta^{\mu\lambda}\sigma^\nu - \eta^{\mu\nu}\sigma^\lambda + i\epsilon^{\mu\nu\lambda\kappa}\sigma_\kappa) \tag{3.55}$$

$$\bar\sigma^{\mu\nu}\bar\sigma^\lambda = \frac{1}{2}(-\eta^{\lambda\nu}\bar\sigma^\mu + \eta^{\lambda\mu}\bar\sigma^\nu - i\epsilon^{\lambda\mu\nu\kappa}\bar\sigma_\kappa) \tag{3.56}$$

$$\bar\sigma^{\mu}\sigma^{\nu\lambda} = \frac{1}{2}(\eta^{\mu\lambda}\bar\sigma^\nu - \eta^{\mu\nu}\bar\sigma^\lambda - i\epsilon^{\mu\nu\lambda\kappa}\bar\sigma_\kappa) \tag{3.57}$$

which imply

$$\sigma^{\mu\nu}\sigma_\nu = \sigma_\nu\bar\sigma^{\nu\mu} = -\frac{3}{2}\sigma^\mu \qquad \bar\sigma^{\mu\nu}\bar\sigma_\nu = \bar\sigma_\nu\sigma^{\nu\mu} = -\frac{3}{2}\bar\sigma^\mu \tag{3.58}$$

Very useful is the version of the previous identities with free spinor indices

$$\sigma^{\mu\nu}_{\alpha}{}^{\beta}\sigma_{\nu\,\gamma\dot\gamma} = \frac{1}{2}(\sigma^{\mu}_{\delta\dot\gamma}\epsilon_{\gamma\alpha}\epsilon^{\beta\delta} - \sigma^{\mu}_{\alpha\dot\gamma}\delta^{\beta}_{\gamma}) \tag{3.59}$$

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\bar{\sigma}_{\nu}^{\dot{\alpha}\gamma} = \frac{1}{2}(\bar{\sigma}^{\mu\,\dot{\alpha}\delta}\epsilon_{\alpha\delta}\epsilon^{\beta\gamma} + \bar{\sigma}^{\mu\,\dot{\alpha}\beta}\delta_{\alpha}^{\gamma}) \tag{3.60}$$

$$\bar{\sigma}^{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\sigma}_{\nu}^{\dot{\gamma}\gamma} = \frac{1}{2}(\bar{\sigma}^{\mu\,\dot{\delta}\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\delta}\dot{\beta}} - \bar{\sigma}^{\mu\,\dot{\alpha}\gamma}\delta_{\dot{\beta}}^{\dot{\gamma}}) \tag{3.61}$$

$$\bar{\sigma}^{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\sigma_{\nu\,\alpha\dot{\gamma}} = \frac{1}{2}(\sigma^{\mu}{}_{\alpha\dot{\delta}}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}} + \sigma^{\mu}{}_{\alpha\dot{\beta}}\delta_{\dot{\gamma}}^{\dot{\alpha}}) \tag{3.62}$$

We also have

$$\sigma^{\mu\nu\beta}_{\alpha}\bar{\sigma}^{\dot{\alpha}}_{\mu\nu\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{\beta}_{\alpha} \tag{3.63}$$

$$\sigma^{0\mu\beta}_{\alpha}\sigma^{\delta}_{\gamma 0\mu} = -\frac{1}{4}(\epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \delta^{\delta}_{\alpha}\delta^{\beta}_{\gamma}) \tag{3.64}$$

$$\bar{\sigma}^{0\mu\dot{\alpha}}_{\dot{\beta}}\bar{\sigma}^{\dot{\gamma}}_{0\mu\dot{\delta}} = -\frac{1}{4}(\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}} + \delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\dot{\gamma}}_{\dot{\beta}}) \tag{3.65}$$

Also useful are the following identities:

$$(\sigma^{\rho\sigma}\sigma^{\mu\nu})^{\alpha}_{\beta}v_{\rho\sigma}v_{\mu\nu} = -\frac{1}{2}\delta^{\alpha}_{\beta}v_{\mu\nu}\hat{v}^{\mu\nu} \qquad (\bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}v_{\rho\sigma}v_{\mu\nu} = -\frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}v_{\mu\nu}\hat{v}^{\dagger\mu\nu} \tag{3.66}$$

$$\begin{aligned} \sigma^{\rho\sigma}\sigma^{\mu}v_{\rho\sigma} &= \hat{v}^{\mu\nu}\sigma_{\nu} \\ \bar{\sigma}^{\mu}\sigma^{\rho\sigma}v_{\rho\sigma} &= -\hat{v}^{\mu\nu}\bar{\sigma}_{\nu} \\ \sigma^{\mu}\bar{\sigma}^{\rho\sigma}v_{\rho\sigma} &= -\hat{v}^{\dagger\mu\nu}\sigma_{\nu} \\ \bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\mu}v_{\rho\sigma} &= \hat{v}^{\dagger\mu\nu}\bar{\sigma}_{\nu} \end{aligned} \tag{3.67}$$

where

$$\hat{v}^{\mu\nu} = v^{\mu\nu} + \frac{i}{2}v^{*\mu\nu} \qquad \hat{v}^{\dagger\mu\nu} = v^{\mu\nu} - \frac{i}{2}v^{*\mu\nu} \tag{3.68}$$

and $v^{\mu\nu} = -v^{\nu\mu}$.

3.3.3 A typical calculation

We present here an example of a typical calculation.
Often we have to reduce expressions of the form

$$\lambda\sigma^{0i}\psi\,\chi\sigma_{0i}\varphi \tag{3.69}$$

In order to do that first we have to write in the spinor indices, then extract the matrices being careful about the position of the spinors involved. Thus the expression above becomes

$$\lambda^{\alpha}\psi_{\beta}\chi^{\gamma}\varphi_{\delta}\;\sigma^{0i\beta}_{\alpha}\sigma^{\delta}_{0i\gamma} \tag{3.70}$$

Then we use the definition (3.37) to write the product of the matrices as

$$\frac{1}{4}(\sigma^0_{\alpha\dot{\alpha}}\bar{\sigma}^{i\dot{\alpha}\beta} - \sigma^i_{\alpha\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\beta})\sigma^{\delta}_{0i\gamma} \tag{3.71}$$

using the properties (3.59) and (3.60) this becomes

$$\frac{1}{8}\Big((\sigma^0\bar{\sigma}_0)_\alpha^{\epsilon}\epsilon_{\gamma\epsilon}\epsilon^{\delta\beta} + (\sigma^0\bar{\sigma}_0)_\alpha^{\delta}\delta_\gamma^{\beta} - (\sigma_0\bar{\sigma}^0)_\epsilon^{\beta}\epsilon_{\alpha\gamma}\epsilon^{\delta\epsilon} + (\sigma_0\bar{\sigma}^0)_\gamma^{\beta}\delta_\alpha^{\delta}\Big) \tag{3.72}$$

using $(\sigma^0\bar{\sigma}_0)_\alpha^\beta = -\delta_\alpha^\beta$ we obtain

$$\frac{1}{8}\Big(- \epsilon_{\gamma\alpha}\epsilon^{\delta\beta} - \delta_\alpha^\delta\delta_\gamma^\beta + \epsilon_{\alpha\gamma}\epsilon^{\delta\beta} - \delta_\gamma^\beta\delta_\alpha^\delta\Big) = -\frac{1}{4}\Big(\epsilon_{\gamma\alpha}\epsilon^{\delta\beta} + \delta_\alpha^\delta\delta_\gamma^\beta\Big) \tag{3.73}$$

When we substitute this back in (3.70), pay attention to the summation conventions and commute the spinors we end up with

$$-\frac{1}{4}(\psi\varphi\,\lambda\chi - \psi\chi\,\lambda\varphi) \tag{3.74}$$

In the case where $\varphi \equiv \lambda$ we can reduce the expression even more using the Fierz identities given in (B.13) in Wess and Bagger. In fact we can write

$$\psi^\alpha\lambda_\alpha\,\lambda^\beta\chi_\beta = -\frac{1}{2}\delta_\alpha^\beta\psi^\alpha\chi_\beta\,\lambda^2 = -\frac{1}{2}\psi\chi\,\lambda^2 \tag{3.75}$$

Thus for $\varphi \equiv \lambda$ the expression (3.70) can be reduced to

$$\frac{3}{8}\psi\chi\,\lambda^2 \tag{3.76}$$

3.3.4 Derivation with respect to a Grassmann variable

The derivative $\frac{\delta}{\delta\psi}$ is a Grassmann variable therefore anti commutes. From the general rule $\partial_\mu = \eta_{\nu\mu}\partial^\nu$ it follows that the indices have to be raised and lowered with the opposite metric tensor with respect to the standard convention

$$\frac{\delta}{\delta\psi^\alpha} = \epsilon_{\beta\alpha}\frac{\delta}{\delta\psi_\beta} \tag{3.77}$$

This is crucial to get the signs right, for instance:

$$\frac{\delta}{\delta\psi^\gamma}(\psi\psi) = \frac{\delta}{\delta\psi^\gamma}(\psi^\alpha\psi_\alpha) = \epsilon_{\alpha\beta}\frac{\delta}{\delta\psi^\gamma}(\psi^\alpha\psi^\beta) = \epsilon_{\alpha\beta}(\delta_\gamma^\alpha\psi^\beta - \psi^\alpha\delta_\gamma^\beta) = +2\psi_\gamma \tag{3.78}$$

and

$$\frac{\delta}{\delta\psi_\gamma}(\psi\psi) = \epsilon^{\beta\gamma}\frac{\delta}{\delta\psi^\beta}(\psi^\alpha\psi_\alpha) = \epsilon^{\beta\gamma}(2\psi_\beta) = -2\psi^\gamma \tag{3.79}$$

Similarly for dotted indices

$$\frac{\delta}{\delta\bar{\psi}_{\dot{\gamma}}}(\bar{\psi}\bar{\psi}) = \frac{\delta}{\delta\bar{\psi}_{\dot{\gamma}}}(\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) = +2\bar{\psi}^{\dot{\gamma}} \quad \frac{\delta}{\delta\bar{\psi}^{\dot{\gamma}}}(\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) = -2\bar{\psi}_{\dot{\gamma}} \tag{3.80}$$

Form here it is clear why the momenta have to be treated with the opposite convention.

3.4 Exercises

Exercise III.a Show that

$$\gamma^5 = \gamma^o\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{3.81}$$

in the Weyl basis.
Exercise III.b Show that

$$[S_m, S_l]^K_L = iC^k_{ml}(S_k)^K_L \tag{3.82}$$

i.e. that the S_ls form a representation of the internal symmetry algebra of the B_ls: $[B_m, B_l] = iC^k_{ml}B_k$.

3.5 Further Reading

[1] J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton University Press, 1992 (Chp 1) (Susy algebra) and [2] (Chp 2) (Irreducible Representations of the Susy algebra)

Chapter 4

Susy Theories

4.1 General Features of Susy Theories

First let us discuss an example were Susy appears in the most simple way: Susy Quantum Mechanics. These results are due to Witten. For a nice account see Ref. [1].

Consider the motion in the direction x of an electron in a magnetic field directed along z but function only of x: $B_z(x)$. The hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q_e\vec{A})^2 + i\frac{q_e}{2m}\vec{\nabla}\cdot\vec{A} + \frac{|q_e|}{2m}\vec{\sigma}\cdot\vec{B}\,, \tag{4.1}$$

where q_e and m are the charge and mass of the electron, respectively, and the natural units are used $\hbar = c = 1$, with the choice

$$A_x = 0 = A_z \quad A_y = \frac{\sqrt{m}}{|q_e|}W(x) \tag{4.2}$$

becomes

$$H = \frac{1}{2}\left(\frac{p^2}{m} + W^2(x) + \sigma_3\frac{1}{\sqrt{m}}\frac{dW}{dx}\right)\,. \tag{4.3}$$

Now define

$$Q_1 \equiv \frac{1}{\sqrt{2}}(\sigma_1\frac{p}{\sqrt{m}} + \sigma_2 W) \quad Q_2 \equiv \frac{1}{\sqrt{2}}(\sigma_2\frac{p}{\sqrt{m}} - \sigma_1 W)\,, \tag{4.4}$$

and discover that Susy is on its way

$$[Q_i, Q_j]_+ = 2\delta_{ij}H \quad [H, Q_i] = 0 \quad i,j = 1,2\,. \tag{4.5}$$

What we have done is to consider "the square root" of the hamiltonian, in the same spirit of how Dirac considered the "square root" of the Klein-Gordon equation $(\Box + \mu^2)\phi = 0$ to obtain his (spinorial) equation $(i \not\partial + \mu)\psi = 0$. There are two important considerations to be done here: i) this toy model Susy shares many of the properties of Susy in field theory; ii) here Susy is implemented in a different spirit respect to the *fundamental* approach of considering it as a symmetry of the S matrix. Here we list other areas of physics where Susy is (or could be) implemented in this spirit:

* Nuclear Physics (Iachello)

* Chaotic Systems (Efetov)

* Superconductivity (Nambu)

* Thermo-Field Dynamics (?)

On a different footing are the following areas, where the Susy scenario is still to be experimentally seen

0 Standard Model

0 Gravity (Supergravity)

0 String Theory

0 Black-Hole Physics

After this brief overview we are ready to discuss the general features of Susy models, based on the algebraic structure of the symmetry itself. The hamiltonian of *any* Susy theory is given in terms of the Susy charges:

$$4H = \bar{\sigma}^{0\alpha\dot{\alpha}}[Q_\alpha, \bar{Q}_{\dot{\alpha}}]_+ = (Q\sigma^0\bar{Q} + \bar{Q}\bar{\sigma}^0 Q) \geq 0 \tag{4.6}$$

on any physical state, where the $N = 1$ case is considered. In particular on the vacuum $|0\rangle$

$$< 0|H|0 >= 0 \quad \text{i.e.} \quad H =: H : \ . \tag{4.7}$$

This can be seen by thinking of Susy as a space time symmetry implemented by the group element of the super-Poincarè group

$$G(x, \theta, \bar{\theta}) = \exp\{i(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})\} \, . \tag{4.8}$$

In gauge theory without SSB the invariance of a model under a given gauge group G is implemented on the action (by construction) and on the vacuum

$$G|0\rangle = e^{\alpha^a T^a}|0\rangle = |0\rangle$$

hence, $T^a|0\rangle = 0$, where T^a, with $a = 1, ..., \dim G$, are the generators of the group. Similarly in a Susy theory

$$Q_\alpha|0\rangle = 0 \quad \bar{Q}_{\dot{\alpha}}|0\rangle = 0\,, \tag{4.9}$$

which proves our statement about the automatic normal ordering of the hamiltonian.
This is a first signal of the nice behavior of Susy theory under renormalization. As a matter of fact, the normal ordering for a quantum field consists in the subtraction of the infinite vacuum energy. For a one-dimensional harmonic oscillator (H.O.)

$$H|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle \to H|0\rangle = \frac{\hbar\omega}{2}|0\rangle$$

this subtraction is finite, and allowed. In (free) field theory the vacuum energy one is discarding with the normal ordering is infinite

$$E_{\text{vaccum}} = \sum_k \frac{\hbar\omega_k}{2} \to \infty\,.$$

Already Pauli in the 1950s realized that in a theory with an equal number of bosons and fermions with equal masses $\omega_k^{\text{bose}} = \sqrt{m^2 + \vec{k}^2} = \omega_k^{\text{fermi}}$ the vacuum energy would have been automatically zero, and not imposed:

$$H = \sum_k \hbar(\omega_k^{\text{bose}} b_k b_k^\dagger + \omega_k^{\text{fermi}} f_k f_k^\dagger) =: H : + \sum_k \hbar(\omega_k^{\text{bose}} - \omega_k^{\text{fermi}}) =: H :\,, \tag{4.10}$$

the minus sign being due to the anti-commuting nature of the fermionic operators $f, f^\dagger$.
The example above given holds for free fields. In fact, the important point here is that this phenomenon holds no matter how complicated is the interaction, even for effective theories, as long as Susy is a symmetry.
It is not surprising then that Susy theories have nice renormalization properties. These properties are exploited in full details in the so-called "non-renormalization theorems" mostly due to Seiberg, even if a better way of calling these theorems would be "no-need-to-renormalize theorems". The idea is again based on the fermi-bose symmetry, which one wants to implement also at the level of the Feynman graphs. For instance, in Susy QED, the vertex $q_e\bar{\psi}\gamma^\mu A_\mu\psi$ has as Susy counterpart $q_e\bar{s\psi}\gamma^\mu A_\mu s\psi$, where $s\psi$ represents the super-partner of the electron ψ within the supermultiplet, hence has the same mass, but bosonic statistics. The fermionic loop is then cancelled by the bosonic one (same coupling q_e and mass).
For instance the perturbative contributions to Quantum SYM theories are

N=1 Tree Level + 1-Loop + ... (well behaved)

N=2 Tree Level + 1-Loop

N=4 Tree Level

hence there are no quantum corrections at all to the N=4 classical SYM! Note that the instanton, non-perturbative, contributions have not been considered.
Note also that in bosonic string theory (the first string theory after Veneziano's hadronic string) the number of space-time (target space) dimensions had to depart from $d = 4$ to $d = 26$ to remove a quantum anomaly. This cancellation takes place, instead, at $d = 10$ when Susy is implemented.
Let us conclude this overview with same nomenclature of the super-partners of the particles within the Susy Standard Model and Gravity:

- fermions: electron, quarks, ... → sfermions: *selectron, squarks, ...*
- gauge bosons → gauginos: *photino, gluino, ...*
- graviton → *gravitino*

the exception is the neutrino which has as super-partner the *neutralino*.

4.2 Wess-Zumino Model

We want to construct now the first example of a four dimensional non-trivial field theory with Susy: the Wess-Zumino model [2]. We shall take a different approach respect to the historical one and introduce the WZ model from first principles (this, of course, can be only done when the results have been already obtained by other means...). This way we will introduce the formalism of $N = 1$ super-space and super-field, which nowadays is widely used by Susy practitioners, and has the advantage of explicitly implementing Susy at all stages of calculations. Nonetheless the final results have to be given in terms of standard (i.e. non-super) fields on standard (Minkowski or Euclidean) space-time.
The space-time nature of Susy is easily seen if we enlarge the space-time to include anti-commuting (Grassman) coordinates

$$x \equiv (x^0, \vec{x}) \to z \equiv (x, \theta, \bar{\theta}) \tag{4.11}$$

where θ_α, and $\bar{\theta}_{\dot{\alpha}}$, are the two-component Weyl spinors, already introduced.

In this *super*-space infinitesimal Susy transformations are generated by

$$\epsilon Q + \bar{\epsilon}\bar{Q} = \epsilon^\alpha(\partial_{\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu) + \bar{\epsilon}_{\dot{\alpha}}(\partial_{\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu)\,. \tag{4.12}$$

Note that these Susy charges close the algebra

$$[Q_\alpha, \bar{Q}_{\dot{\alpha}}]_+ = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \quad [Q, Q]_+ = 0 = [\bar{Q}, \bar{Q}]_+\,. \tag{4.13}$$

Let us now introduce the most general field defined on z and expand it in powers of θ and $\bar{\theta}$

$$\begin{aligned} F(x,\theta,\bar{\theta}) \;\; = \;\; & \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) \\ & +\theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ & +\theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\xi(x) + \bar{\theta}\bar{\theta}\theta\theta d(x) \end{aligned} \tag{4.14}$$

where $\phi(x), m(x), n(x), v_\mu(x), d(x)$ and $\psi(x), \bar{\chi}(x), \bar{\lambda}(x), \xi(x)$ are bosonic and fermionic fields, respectively, whose properties under Lorentz transformations are fixed by those of $\theta, \bar{\theta}, \sigma^\mu$, and by the requiring $F(z)$ to be a scalar under Lorentz.
The Susy transformations of the component fields can be obtained by noticing that

$$(\epsilon Q + \bar{\epsilon}\bar{Q}) \cdot F \equiv \delta_\epsilon F \tag{4.15}$$

where the infinitesimal action of the charges is given in Eq. (4.12). Note that this procedure, while quite straightforward, hides the difficulties of implementing Susy as a Noether symmetry. On this see the last lecture.
One can define the super-covariant derivatives $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot{\alpha}}$

$$[\mathcal{D}_\alpha, Q_\beta]_+ = [\mathcal{D}_\alpha, \bar{Q}_{\dot{\beta}}]_+ = [\bar{\mathcal{D}}_{\dot{\alpha}}, Q_\beta]_+ = [\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}]_+ = 0 \tag{4.16}$$

in the following way

$$\mathcal{D}_\alpha = \partial_{\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \quad \text{and} \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\partial_{\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\,. \tag{4.17}$$

We can now define the *chiral* super-field Φ as the one satisfying the following constraint in super-space

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0\,, \tag{4.18}$$

while the other chirality is obtained by the constraint $\mathcal{D}_\alpha\Phi^\dagger = 0$. The component fields are obtained as in the general case, but the constraint reduces the number of component

$$\begin{aligned} \Phi(z) \;\; = \;\; & \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) \\ & \frac{-i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\bar{\theta}\bar{\theta}\theta\theta\Box\phi(x) \end{aligned}$$

and only the left chiral ψ_α is present. These are the WZ fields: ϕ the dynamical bosonic field, ψ the dynamical fermionic field, F the dummy (lagrange multiplier) bosonic field. Of course the component fields of $\Phi^\dagger$ are $\phi^\dagger, \bar\psi_{\dot\alpha}, F^\dagger$, with the right chiral spinor.
Super-fields, then, explicitly contain all the members of a given super-multiplet, hence under Susy its components have to transform into one another. The general rule is that the component in the position n transforms into combination (and derivatives) of the components in positions n+1 and n-1. The last term transforms into the second last, which is always a pure divergence. As will be fully explained in the last lecture, in Susy the lagrangian densities transform into a pure divergence $\delta_\epsilon \mathcal{L} = \partial_\mu V^\mu$. It is then natural to identify the last term in the expansion of super-field products as the lagrangian density of the theory. For the WZ model

$$\begin{aligned} \mathcal{L}_{WZ} &= \Phi^\dagger\Phi|_{\bar\theta\bar\theta\theta\theta} + [(\frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3)_{\theta\theta} + (\text{h.c.})] \\ &= \int d^2\bar\theta d^2\theta \Phi^\dagger\Phi + [\int d^2\theta(\frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3) + (\text{h.c.})] \end{aligned} \tag{4.19}$$

which gives the interactive, massive WZ model

$$\begin{aligned} \mathcal{L}_{WZ} &= -\frac{i}{2}\psi\,\partial\!\!\!/\bar\psi - \frac{i}{2}\bar\psi\,\bar{\partial\!\!\!/}\psi - \partial_\mu\phi\partial^\mu\phi^\dagger + m^2\phi\phi^\dagger - \frac{m}{2}\psi^2 - \frac{m}{2}\bar\psi^2 \\ &\quad -g(\psi\psi\phi + \bar\psi\bar\psi\phi^\dagger) + mg(\phi^\dagger\phi^2 + \phi\phi^{\dagger 2}) + g\phi^2\phi^{\dagger 2} \end{aligned} \tag{4.20}$$

where the dummy fields have been eliminated through their Euler-Lagrange expressions.

4.3 Exercises

Exercise IV.a Prove that in Susy Quantum Mechanics the Hamiltonian is positive *semi* definite.
Exercise IV.b The Lagrangian density and Susy transformations of the fields for the WZ massive model are given by

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi^\dagger + FF^\dagger + [(-\frac{i}{2}\psi\,\partial\!\!\!/\bar\psi + \frac{m}{2}\phi F - \frac{m}{2}\psi^2) + (\text{h.c.})] \tag{4.21}$$

and

$$\delta\phi = \sqrt{2}\epsilon\psi \qquad \delta\phi^\dagger = \sqrt{2}\bar\epsilon\bar\psi \tag{4.22}$$

$$\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar\epsilon)_\alpha\partial_\mu\phi + \sqrt{2}\epsilon_\alpha F \qquad \delta\bar\psi^{\dot\alpha} = i\sqrt{2}(\bar\sigma^\mu\epsilon)^{\dot\alpha}\partial_\mu\phi^\dagger + \sqrt{2}\bar\epsilon^{\dot\alpha}F^\dagger \tag{4.23}$$

$$\delta F = i\sqrt{2}\bar\epsilon\,\bar{\partial\!\!\!/}\psi \qquad \delta F^\dagger = i\sqrt{2}\epsilon\,\partial\!\!\!/\bar\psi \tag{4.24}$$

where ϕ is a complex scalar field, ψ is its Susy fermionic partner in Weyl notation and we kept the complex bosonic dummy field F in the lagrangian density. Compute the Susy-Noether current for this model and give a recipe to transform the dummy field F under the Susy-Noether charge.

4.4 Further Reading

[1] M.A.Shifman, *ITEP Lectures on Particle Physics and Field Theory*, World Scientific 1999 (Vol I, Chp IV); [2] J.Wess, B.Zumino, Phys. Lett. B 49 (1974) 52.

Chapter 5

Seiberg-Witten Theory I

5.1 N=2 Susy Classical Georgi-Glashow model and the Mass Formula

In [1] the authors considered the classical N=2 supersymmetric Georgi-Glashow Action with gauge group O(3) spontaneously broken down to U(1) and its supercharges. For the sake of consistency we shall instead consider the gauge group SU(2), since all the considerations are the same.
The fastest, and explicitly Susy, way of introducing the N=2 SYM action is to present the N=2 superfield Ψ^a, where $a = 1, 2, 3$ is the gauge group index, SU(2) in our case. This N=2 superfield contains two N=1 superfields related by R-symmetry: the chiral multiplet $\Phi^a \equiv (\phi^a, \psi^a_\alpha, F^a)$ (which we already discussed) and the vector multiplet $V^a \equiv (\lambda^a_\alpha, A^a_\mu, D^a)$, where, for each a, ϕ is the scalar field, ψ, λ are two Weyl fermionic fields, A_μ is a vector field, and F, D are bosonic dummy fields. Note that in the following we shall slightly change the notation and will call E the dummy field F. The N=1 vector field is obtained by imposing the reality condition in superspace

$$V^\dagger = V$$

and the WZ gauge which sets to zero (gauges away) a certain number of component fields. The Susy transformations of the component fields are given by

first Susy, parameter ϵ_1

$$\begin{aligned} \delta_1 \vec{\phi} &= \sqrt{2}\epsilon_1 \vec{\psi} \\ \delta_1 \vec{\psi}^\alpha &= \sqrt{2}\epsilon_1^\alpha \vec{E} \\ \delta_1 \vec{E} &= 0 \end{aligned} \tag{5.1}$$

$$
\begin{aligned}
\delta_1 \vec{E}^\dagger &= i\sqrt{2}\epsilon_1 \not{\mathcal{D}}\vec{\bar{\psi}} + 2i[\vec{\phi}^\dagger, \epsilon_1 \vec{\lambda}] \\
\delta_1 \vec{\bar{\psi}}_{\dot{\alpha}} &= -i\sqrt{2}\epsilon_1^\alpha \not{\mathcal{D}}_{\alpha\dot{\alpha}} \vec{\phi}^\dagger \\
\delta_1 \vec{\phi}^\dagger &= 0
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
\delta_1 \vec{\lambda}^\alpha &= -\epsilon_1^\beta (\sigma^{\mu\nu}{}_\beta{}^\alpha \vec{F}_{\mu\nu} - i\delta_\beta^\alpha \vec{D}) \\
\delta_1 \vec{A}^\mu &= i\epsilon_1 \sigma^\mu \vec{\bar{\lambda}} \quad \delta_1 \vec{D} = -\epsilon_1 \not{\mathcal{D}} \vec{\bar{\lambda}} \\
\delta_1 \vec{\bar{\lambda}}_{\dot{\alpha}} &= 0
\end{aligned} \tag{5.3}
$$

second Susy, parameter ϵ_2

$$
\begin{aligned}
\delta_2 \vec{\phi} &= -\sqrt{2}\epsilon_2 \vec{\lambda} \\
\delta_2 \vec{\lambda}^\alpha &= -\sqrt{2}\epsilon_2^\alpha \vec{E}^\dagger \\
\delta_2 \vec{E}^\dagger &= 0
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
\delta_2 \vec{E} &= -i\sqrt{2}\epsilon_2 \not{\mathcal{D}}\vec{\bar{\lambda}} + 2i[\vec{\phi}^\dagger, \epsilon_2 \vec{\psi}] \\
\delta_2 \vec{\bar{\lambda}}_{\dot{\alpha}} &= i\sqrt{2}\epsilon_2^\alpha \not{\mathcal{D}}_{\alpha\dot{\alpha}} \vec{\phi}^\dagger \\
\delta_2 \vec{\phi}^\dagger &= 0
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
\delta_2 \vec{\psi}^\alpha &= -\epsilon_2^\beta (\sigma^{\mu\nu}{}_\beta{}^\alpha \vec{F}_{\mu\nu} + i\delta_\beta^\alpha \vec{D}) \\
\delta_2 \vec{A}^\mu &= i\epsilon_2 \sigma^\mu \vec{\bar{\psi}} \quad \delta_2 \vec{D} = \epsilon_2 \not{\mathcal{D}} \vec{\bar{\psi}} \\
\delta_2 \vec{\bar{\psi}}_{\dot{\alpha}} &= 0
\end{aligned} \tag{5.6}
$$

Where the generic SU(2) vector is defined as $\vec{X} = \frac{1}{2}\sigma^a X^a$ with $a = 1, 2, 3$ and we follow the summation convention. The covariant derivative and the vector field strength are given by

$$\mathcal{D}_\mu X^a = \partial_\mu X^a + \epsilon^{abc} A_\mu^b X^c \tag{5.7}$$

and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \tag{5.8}$$

The action for the SU(2) N=2 SYM theory in N=2 superspace is simply

$$\mathcal{A} = \frac{1}{4\pi} \mathrm{Im} \int d^4x d^2\theta d^2\tilde{\theta} \frac{1}{2} \tau \Psi^a \Psi^a \tag{5.9}$$

where θ and $\tilde{\theta}$ are the grassmanian coordinates of the N=2 superspace and

$$\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2}\,, \tag{5.10}$$

is the *complex* coupling constant whose real and imaginary part are related to the CP violating ϑ-angle of the ϑ-vacua, and to the SU(2) coupling, respectively.
By performing the expansion in the Grassman variables the action becomes

$$\begin{aligned}\mathcal{A} &= \int d^4x\{-\frac{1}{g^2}[\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mathcal{D}_\mu\phi^a\mathcal{D}^\mu\phi^{a\dagger} + i\psi^a\ \not{\mathcal{D}}\bar{\psi}^a + i\lambda^a\ \not{\mathcal{D}}\bar{\lambda}^a\,, \\ &+ \frac{1}{\sqrt{2}}(\psi^a[\lambda,\phi]^a + \text{h.c.}) - \frac{1}{2}([\phi,\phi^\dagger]^a[\phi,\phi^\dagger]^a)] - \frac{\vartheta}{64\pi^2}F^a_{\mu\nu}F^{*a\mu\nu}\}\,.\end{aligned}$$

Note that the last term is the instanton term. This is a pure divergence of the form $\partial_\mu K^\mu$, where

$$K^\mu \sim \epsilon^{\mu\nu\rho\sigma}(A^a_\nu\partial_\rho A^a_\sigma + \epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma)$$

and when integrated in the action gives the Pontryagin index. Note also, that due to the pseudo-tensor (axial) nature of the dual $F^*_{\mu\nu}$, the time-reversal symmetry is broken. That is why the ϑ-angle is related to CP violation. On this point see, for instance, Ryder's last chapter, and Weinberg's first (discrete symmetries C, P, T) and second (topological objects in QFT, in particular the QCD ϑ-vacua) volumes.
The central charge for this theory was obtained in [1]

$$Z = i\sqrt{2}\int d^2\vec{\Sigma}\cdot(\vec{\Pi}^a\phi^a + \frac{1}{4\pi}\vec{B}^a\phi^a_D) \qquad a = 1,2,3 \tag{5.11}$$

where $d^2\vec{\Sigma}$ is the measure on the sphere at spatial infinity S^2_∞, the ϕ^a's are the scalar fields, the $\vec{B}^a$'s are the magnetic fields, $\vec{\Pi}^a$ is the conjugate momentum of the vector field $\vec{A}^a$ and $\phi^a_D \equiv \tau\phi^a$. In the classical case ϕ^a_D is merely a formal quantity with no precise physical meaning. On the contrary, in the low-energy sector of the quantum theory, it becomes the *e.m. dual* of the scalar field.
In the unbroken phase $Z = 0$, but, as well known, in the broken phase this theory admits 't Hooft-Polyakov monopole solutions. In this phase the scalar fields (and the vector potentials) tend to their vacuum value $\phi^a \sim a\frac{r^a}{r}$ ($A^{ai} \sim \epsilon^{iab}\frac{r^b}{r^2}$, $A^{a0} = 0$), where $a \in \mathbf{C}$, as $r \to \infty$. This behavior gives rise to a magnetic charge. By performing a SU(2) gauge transformation on this radially symmetric ("hedgehog") solution we can align $< 0|\phi^a|0 >$ along one

direction (the Coulomb branch), say $< 0|\phi^a|0 >= \delta^{a3}a$, and the 't Hooft-Polyakov monopole becomes a U(1) Dirac-type monopole.
In this spirit we can define the electric and magnetic charges as

$$q_e \equiv \frac{1}{a}\int d^2\vec{\Sigma}\cdot\vec{\Pi}^3\phi^3 \tag{5.12}$$

$$q_m \equiv \frac{1}{a}\int d^2\vec{\Sigma}\cdot\frac{1}{4\pi}\vec{B}^3\phi^3 = \frac{1}{a_D}\int d^2\vec{\Sigma}\cdot\frac{1}{4\pi}\vec{B}^3\phi^3_D \tag{5.13}$$

where $a_D = \tau a$ and only the U(1) fields remaining massless after SSB appear. These quantities are quantized, since[1] $q_m \in \pi_1(U(1)) \sim \pi_2(SU(2)/U(1)) \sim \mathbf{Z}$ and q_e is quantized due to Dirac quantization of the electric charge in presence of a magnetic charge.
Thus, after SSB, the central charge becomes

$$Z = i\sqrt{2}(n_e a + n_m a_D) \tag{5.14}$$

The mass spectrum of the theory is then given by

$$M = \sqrt{2}|n_e a + n_m a_D| \tag{5.15}$$

We shall call this formula the Montonen-Olive mass formula. It is now crucial to notice that this formula holds for the whole spectrum consisting of *elementary* particles, two W bosons and two fermions, and *topological* excitations, monopoles and dyons. For instance the mass of the W bosons and the two fermions can be obtained by setting $n_e = \pm 1$ and $n_m = 0$, which gives $m_W = m_{\text{fermi}} = \sqrt{2}|a|$, whereas the mass of a monopole ($n_e = 0$ and $n_m = \pm 1$) amounts to $m_{\text{mon.}} = \sqrt{2}|a_D|$. This establishes a democracy between particles and topological excitations that becomes more clear when e.m. duality is implemented.

In our short-cut to write down the classical version of the mass formula, we did not follow the chronological order of the various discoveries that led to it. First Bogomolnyi, Prasad and Sommerfield showed that, for a theory admitting monopole solutions, the formula

$$M = a(q_e^2 + q_m^2)^{1/2} \tag{5.16}$$

holds classically for monopoles and dyons (topological excitations carrying electric and magnetic charge). Then Montonen and Olive showed that it

[1] We say that the 't Hooft-Polyakov magnetic charge is the winding number of the map $SU(2)/U(1) \sim S^2 \to S^2_\infty$, that identifies the homotopy class of the map. By considering the maps $U(1) \sim S^1 \to S^1_\infty$, where S^1_∞ is the equator of S^2_∞, it is clear that a similar comment holds for the U(1) Dirac-type magnetic charge. It turns out that the two homotopy groups, $\pi_2(S^2)$ and $\pi_1(S^1)$, are isomorphic to $\mathbf{Z}$.

is true classically for all the states, elementary particles included. Finally Witten and Olive [1] obtained it, again classically, from the N=2 Susy. The formula (5.16) can be written in the following form

$$M = |ag(n_e + \tau_0 n_m)| \tag{5.17}$$

where $q_e \equiv gn_e$, $q_m = (-4\pi/g)n_m$ and $\tau_0 \equiv i4\pi/g^2$. This is the formula we are showing here, provided $ag \to a$ and τ_0 is *improved* to τ.

5.2 E-M Duality Revisited

As explained above e-m duality is not a symmetry in the Noether sense, but rather a transformation that connects different regimes. To see how this applies to our problem let us look again at the central charge Z

$$Z = i\sqrt{2}(n_e a + n_m a_D) = i\sqrt{2}(n_m, n_e) \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{5.18}$$

If we act with $S^{-1} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the row vector (n_m, n_e), we exchange electric charge with magnetic charge and *vice-versa*. This is the e.m. duality transformation: it maps electrically charged elementary particles to magnetically charged collective excitations, giving meaning to the democracy announced above between all the BPS states.
The mass of all the particles has to be given by the mass formula (5.15). Therefore to S^{-1} acting on (n_m, n_e) it has to correspond S acting on the column vector, namely

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to S \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} a \\ -a_D \end{pmatrix} \tag{5.19}$$

so that Z is left invariant. The mass formula is actually invariant under the full group $SL(2, \mathbf{Z})$ of 2×2 unimodular matrices with integer entries, generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{5.20}$$

where $b \in \mathbf{Z}$.

5.3 Quantum Theory

The N=2 supersymmetric, SU(2) gauged, Wilsonian effective Action[2] in N=2 superfield language is given by

$$\mathcal{A} = \frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta d^2\tilde{\theta}\mathcal{F}(\Psi^a\Psi^a) \tag{5.21}$$

where θ and $\tilde{\theta}$ are the grassmanian coordinates of the N=2 superspace and $\Psi^a\Psi^a$, $a = 1, 2, 3$, is the SU(2) gauge Casimir. Ψ^a is the N=2 superfield that combines a scalar field ϕ^a, a vector field A^a_μ, two Weyl fermions ψ^a and λ^a (and possibly dummy fields) into a single Susy multiplet. Thus all the fields are in the same representation of the gauge group SU(2) as A^a_μ, i.e. the adjoint representation. $\mathcal{F}$ is a holomorphic[3] and analytic[4] function.
The point we want to make here is that the knowledge of the function $\mathcal{F}$, sometimes called *prepotential*, completely determines the theory.
The N=2 Yang-Mills low-energy U(1) effective Lagrangian for the light modes only, up to second derivatives of the fields and four fermions is given by

$$\begin{aligned}\mathcal{L} &= \frac{\mathrm{Im}}{4\pi}\int d^4x\Big(-\mathcal{F}''(\phi)[\partial_\mu\phi^\dagger\partial^\mu\phi + \frac{1}{4}F_{\mu\nu}\hat{F}^{\mu\nu} + i\psi\ \partial\!\!\!/\bar{\psi} + i\lambda\ \partial\!\!\!/\bar{\lambda} - (EE^\dagger + \frac{1}{2}D^2)] \\ &+ \mathcal{F}'''(\phi)[\frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}\psi F_{\mu\nu} - \frac{1}{2}(E^\dagger\psi^2 + E\lambda^2) + \frac{i}{\sqrt{2}}D\psi\lambda] + \mathcal{F}''''(\phi)[\frac{1}{4}\psi^2\lambda^2]\Big)\end{aligned} \tag{5.22}$$

where the primes denote derivation with respect to the scalar field ϕ: $\mathcal{F}'' = \partial^2\mathcal{F}/\partial\phi^2$, etc.. For large values of a ($a >> \Lambda$) the theory is weakly coupled, thus a perturbative computation to evaluate $\mathcal{F}$ leads to a reliable result. This computation was performed by Seiberg and it turns out that

$$\mathcal{F}(\phi) = \frac{1}{2}\tau\phi^2 + \phi^2\frac{i\hbar}{2\pi}\ln(\frac{\phi^2}{\Lambda^2}) + \phi^2\sum_{k=1}^{\infty} c_k\frac{\Lambda^{4k}}{\phi^{4k}} \tag{5.23}$$

---

[2]The Wilsonian effective Action differs from the standard one particle irreducible effective Action when massive and massless modes are both present. The Wilsonian effective Action allows for the description of the strong coupling regimes in terms of massless (or light) modes only. We shall not enter into details here.

[3]$\mathcal{F}$ is not a function of $\bar{\Psi}$ and this only happens if we stop at the leading order term in the expansion in p_μ of the effective Action. For instance the next-to-leading order term $\mathcal{H}(\Psi, \bar{\Psi})$ is no longer holomorphic.

[4]By analytic, we mean that it can have branch cuts, poles etc., but no essential singularities.

where the function is parameterized by the only massless scalar field surviving along the Coulomb branch. The first two terms are the perturbative contributions: tree level and one loop terms respectively[5], and the last term is the non-perturbative instanton contribution. From this expression we see that the classical limit consists in the substitution $\mathcal{F}(\phi) \rightarrow \frac{1}{2}\tau\phi^2$, and the effective complex coupling is $\mathcal{F}''(\phi) = \mathcal{R} + i\mathcal{I}$, while its imaginary part $\mathcal{I}$ plays the role of (the inverse of the square of) the gauge group coupling g. What happens to the mass formula (5.15)in the quantum phase? Seiberg and Witten conjectured that, due to the preservation of the N=2 Susy, the formula is *formally* unchanged: quantum corrections play a major role since now

$$a_D^{\text{class}} = \tau_{\text{class}} a \rightarrow a_D^{\text{eff}} \equiv \frac{\partial \mathcal{F}(a)}{\partial a} \tag{5.24}$$

where $\mathcal{F}(a)$ is the prepotential in the low energy sector, evaluated at a, but no other changes are expected. Therefore the quantum improvement of the Montonen-Olive mass formula (5.15) is simply given by

$$M = \sqrt{2}|n_e a + n_m \mathcal{F}'(a)| \tag{5.25}$$

This statement is vital for the whole theory. Nevertheless no direct proof from the N=2 Susy algebra was presented in the original paper.

5.4 Exercises

Exercise V.a It is a very deep result that the masses in these models are given by the non-trivial combination of v.e.v.'s of the Higgs field, and electric and magnetic charges. Probably the most important consequence of this occurrence is the fact that *all* the excitations, elementary and topological, obey the same formula. Why is this so important?
Exercise V.b S duality maps electrically charged particles into magnetically charged monopoles. What is the effect of the T duality transformation?

5.5 Further Reading

[1] E.Witten, D.Olive, Phys. Lett. B 78 (1978) 97; [2] A. Bilal, *Duality in N=2 Susy SU(2) Yang-Mills Theory: a pedagogical introduction to the work of Seiberg and Witten*, hep-th/9601007.

[5]For N=2 Susy these are the only two contributions to the perturbative $\mathcal{F}$ (non-renormalization) whereas for N=4 the tree level (classical) term is enough (super-renormalization).

Chapter 6

Seiberg-Witten Theory II

6.1 The Reduction to a Riemann-Hilbert Problem

The key idea of Seiberg and Witten is to compute $\mathcal{F}$ by first posing and then solving what mathematicians call a "Riemann-Hilbert (RH) problem", namely: *given as initial data singularities and monodromies, does there exist a Fuchsian system having these data*? In a paper published in 1900 Hilbert presented a list of 23 problems. The statement we are describing here appears as the 21^{st} one in the list. The RH problem seems to be very fruitful in physics. Recently it has been applied to renormalization in Quantum Field Theory.

A Fuchsian system is a system of differential equations in the complex domain, given by

$$\frac{df^i(z)}{dz} = A^{ij}(z)f^j(z) \quad i,j = 1,...,p \tag{6.1}$$

where the $f^i(z)$'s are in general multi-valued complex functions and the matrix $A(z)$ is holomorphic in $S = \mathbf{C} - \{z_1,...,z_n\}$ and $z_1,...,z_n$ are poles of $A(z)$ of order at most one. We can naturally associate a group structure to a fundamental system of solutions of (6.1), say $GL(p,\mathbf{C})$. This corresponds to the simple request to have p linearly independent solutions combined together into an invertible $p \times p$ complex matrix, say $F(z) \in GL(p,\mathbf{C})$. We shall see that in SW theory this group turns out to be a subgroup of $SL(2,\mathbf{Z})$, namely

$$\Gamma_2 \equiv \left\{\gamma \in \mathrm{SL}(2,\mathbf{Z}) : \gamma = \mathbf{1} + \begin{pmatrix} l & m \\ n & p \end{pmatrix} l,m,n,p \in \mathbf{Z}\right\} \tag{6.2}$$

If we now consider the universal covering surface of S, say $\tilde{S}$, we can define maps $\delta : \tilde{S} \to S$. This simply means that we are considering all the Riemann

sheets obtained by winding around the singularities $z_1, ..., z_n$. For instance, in the case of a logarithmic function of one complex variable, $\tilde{S}$ represents the infinite copies of the complex plane. The *monodromy* representation of $GL(p, \mathbf{C})$ is then defined as $M : \delta \to M(\delta) \in GL(p, \mathbf{C})$. More practically the monodromy constant matrices M are obtained by winding around the singularities z_i's of $A(z)$ with loops α_i's in one-to-one correspondence with the z_i's.
Therefore the RH problem consists in finding a system of the type (6.1) starting from the knowledge of the singularities $z_1, ..., z_n$ and the monodromies around them.
We want to show in the following how these singularities arise in SW model, their physical meaning and the vital role of the central charge Z of the underlying N=2 Susy.

6.2 Singularities, Monodromies and Space of Gauge-Inequivalent Vacua (Moduli)

Since the Higgs v.e.v. $a \in \mathbf{C}$, we can think of $\mathbf{C}$ as the space of gauge inequivalent vacua, namely to a, a', with $a \neq a'$, correspond two vacua not related by a SU(2) gauge transformation (but only by a transformation in the little group U(1)). To be more precise we have to introduce the SU(2) invariant parameter

$$u(a) = \mathrm{Tr} < 0|\vec{\phi}^2|0 >= \frac{1}{2}a^2 \tag{6.3}$$

to get rid of the ambiguity due to the discrete Weyl group of SU(2), which still acts as $a \to -a$ within the Cartan subalgebra. This is now a good coordinate on the complex manifold of gauge inequivalent vacua. We shall call this manifold $\mathcal{M}$, for *moduli* space.
Eventually we can define a singularity of $\mathcal{M}$ as *a value of u at which some of the particles of the spectrum, either elementary or topological, become massless.* Classically there is only one of such values, namely $u = 0$ where the SU(2) gauge symmetry is fully restored and $\mathcal{M}$ looses its meaning. It is worthwhile to notice that the classical moduli space is merely a tool to introduce the idea of a singularity, since the running of the coupling is a purely quantum effect, therefore there is no physical reason to vary u classically. Nevertheless the crucial point is to keep the idea of a singularity of $\mathcal{M}$ as a point where *some* particles become massless. The big step is to go to the quantum theory where non trivial renormalization leads to a non vanishing beta function. Seiberg and Witten conjectured that, in

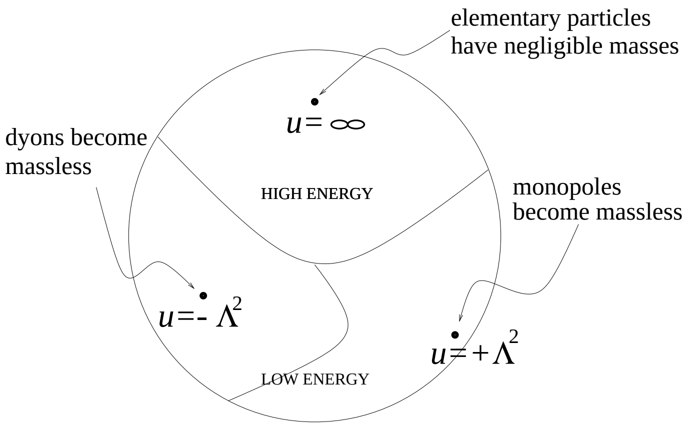

Figure 6.1: The quantum moduli space $\mathcal{M}_q$. The singularities and the different corresponding *phases* are shown.

the quantum theory, the singularity at $u = 0$ splits into $u = \pm\Lambda^2$, where monopoles and dyons, and *not* W bosons and fermions (as in the classical case) are supposed to become massless (see Figure).

The vital importance of the mass formula is immediately seen if one wants to define the singularities of the quantum theory in the spirit outlined above. In fact Seiberg and Witten conjectured that, in the quantum theory, the singularity at $u = 0$ splits into $u = \pm\Lambda^2$, where monopoles and dyons, and *not* W bosons and fermions (as in the classical case) are supposed to become massless. This makes sense if the W bosons in the low energy sector can decay into a monopole + dyon pair. Since all the states are BPS, one can show that the mass formula (5.25) indeed allows for this decay. Thus, if some particles have to become massless in the low energy sector, these cannot be the W bosons, whose mass is frozen at low energies, but only the topological excitations. Of course this is only a sufficient but not necessary condition for this to happen. Furthermore one should explain why only two singularities and why at $\pm\Lambda^2$ and there is no rigorous proof of these points. Quantum corrections play a major role since now the e-m dual of the Higgs field is

$$a_D^{\rm eff} \equiv \frac{\partial \mathcal{F}(a)}{\partial a} \tag{6.4}$$

where $\mathcal{F}(a)$ is the prepotential in the low energy sector, evaluated at a, but no other changes are expected. Therefore the quantum improvement of the classical mass formula is given by

$$M = \sqrt{2}|n_e a + n_m \mathcal{F}'(a)| \tag{6.5}$$

This statement is vital for the whole theory. Let us accept this expression for the moment. We shall later prove it.
The monodromy at $u = \infty$ can be easily computed, since here we can trust the perturbative expansion of $\mathcal{F}$ evaluated at a. We have $a = \sqrt{2u}$ and $a_D(u) = \mathcal{F}'(u) \sim i\frac{\hbar}{\pi}\sqrt{2u}\Big(\ln(\frac{2u}{\Lambda^2}) + 1\Big)$. By winding around $u = \infty$, the branch point of the logarithm, we obtain

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to \begin{pmatrix} a_D(e^{i2\pi}u) \\ a(e^{i2\pi}u) \end{pmatrix} = \begin{pmatrix} -a_D(u) + 2a(u) \\ -a(u) \end{pmatrix} \tag{6.6}$$

or $\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix}$ where

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{6.7}$$

To find the other two monodromies we require the state of vanishing mass responsible for the singularity to be invariant under the monodromy transformation:

$$(n_m, n_e)M_{(n_m,n_e)} = (n_m, n_e) \tag{6.8}$$

This simply means that, even if SL(2, $\mathbf{Z}$) maps particles of one phase to particles of another phase, once we arrive at a singularity the monodromy does not change this state into another state. From this it is easy to check that the form of the monodromies around the other two points $\pm\Lambda^2$ has to be

$$M_{(n_m,n_e)} = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} \tag{6.9}$$

Note that M_∞ is not of this form.
The global consistency conditions on how to patch together the local data is simply given by

$$M_{+\Lambda^2} \cdot M_{-\Lambda^2} = M_\infty \tag{6.10}$$

which follows from the fact that the loops around $\pm\Lambda^2$ can be smoothly pull around the Riemann sphere to give the loop at infinity.
In the Ising model the *gluing* of the different local data consists in the identification of a self-dual point $K = K^*$, where $K = J/k_B T << 1$ is the

coupling at high temperature T and $K^* >> 1$ is the coupling at low temperature given by $\sinh(2K^*) = (\sinh(2K))^{-1}$, J is the strength of the interaction between nearest neighbors and k_B the Boltzman constant. This determines exactly the critical temperature of the phase transition T_c given by $\sinh(2J/k_BT_c) = 1$.
By using the expression (6.9) we can obtain the solution of this equation given by

$$M_{+\Lambda^2} = M_{(1,0)} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad M_{-\Lambda^2} = M_{(1,1)} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{6.11}$$

and we see that the particles becoming massless are indeed monopoles and dyons as conjectured.
The monodromy matrices generate the subgroup Γ_2 of the full duality group $SL(2, \mathbf{Z})$ given in (6.2).

6.3 The Solution

We have now all the ingredients and we can write down the announced Fuchsian equation

$$\frac{d^2 f(z)}{dz^2} = A(z) f(z) \tag{6.12}$$

where

$$A(z) = -\frac{1}{4}\Big[\frac{1-\lambda_1^2}{(z+1)^2} + \frac{1-\lambda_2^2}{(z-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(z+1)(z-1)}\Big] \tag{6.13}$$

$z \equiv u/\Lambda^2$ and $A(z)$ exhibits the described singularities at $z = \pm 1$ and $z = \infty$. Note that his is a second order differential equation therefore it is equivalent to a Fuchsian system (6.1) with $p = 2$, the singularity at $z = \infty$ being on a somehow different footing. Note also that the poles of $A(z)$ become second order.
To clarify the physical meaning of this approach one has to think of Eq. (6.12) as a one-dimensional Schrödinger-like equation, with $A(z)$ playing the role of a potential $V(x)$, and $f(z)$ of the wave function $\psi(x)$

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = 0\,, \tag{6.14}$$

where we shall deal with functions of real variables for a moment only, and the potential is supposed to be periodic $V(x + 2\pi) = V(x)$. The last differential equation has, of course, two independent solutions $\psi_1(x)$ and $\psi_2(x)$. Due to the periodic potential the differential equation is exactly the same at x and $x+2\pi$, hence one has to find that the solutions at $x+2\pi$ are linear combinations of the solutions at x:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (x+2\pi) = M \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (x) \,, \tag{6.15}$$

where M is the constant monodromy matrix.
Seiberg and Witten have found that the coefficients are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$, thus

$$A(z) = -\frac{1}{4}\frac{1}{(z+1)(z-1)} \tag{6.16}$$

The two solutions of (6.12) with $A(z)$ in (6.16), are given in terms of hypergeometric functions:

$$f_1(z) \equiv a_D(z) = i\frac{z-1}{2}F\left(\frac{1}{2},\frac{1}{2},2;\frac{1-z}{2}\right)\,, \tag{6.17}$$

$$f_2(z) \equiv a(z) = \sqrt{2(z+1)}F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{z+1}\right)\,, \tag{6.18}$$

or, by using the integral representation of the hypergeometric functions

$$a_D(z) = \frac{\sqrt{2}}{\pi}\int_1^z dx\frac{\sqrt{x-z}}{\sqrt{x^2-1}} \tag{6.19}$$

$$a(z) = \frac{\sqrt{2}}{\pi}\int_{-1}^1 dx\frac{\sqrt{x-z}}{\sqrt{x^2-1}}\,. \tag{6.20}$$

We can invert Eq. (6.18) to obtain $z(a)$ then substitute this into $a_D(z)$ to obtain

$$a_D(a) = \frac{d}{da}\mathcal{F}(a)\,.$$

Integrating with respect to a yields to $\mathcal{F}(a)$: the theory is solved!
As noted above this expression of $\mathcal{F}(a)$ is not globally valid on $\mathcal{M}_q$, but only near $u=\infty$. For the other two regions one has $\mathcal{F}_D(a_D)$ near $u=+\Lambda^2$ and $\mathcal{F}_D(a-2a_D)$ near $u=-\Lambda^2$.

6.4 Exercises

Exercise VI.a Give a qualitative argument why $\mathcal{F}(a)$ is not the right quantity to consider at low energies, i.e. near the quantum singularities of $\mathcal{M}_q$ at $\pm\Lambda^2$.

Exercise VI.b By using the expression (6.9) of the monodromies around the quantum singularities $\pm\Lambda^2$, find the solutions (6.11) of the global consistency conditions (6.10).

6.5 Further Reading

[1] N.Seiberg and E.Witten, Nucl. Phys. B 426 (1994) 19; [2] A. Bilal, *Duality in N=2 Susy SU(2) Yang-Mills Theory: a pedagogical introduction to the work of Seiberg and Witten*, hep-th/9601007.

Chapter 7

Susy-Noether Currents and SW Central Charge

7.1 Susy-Noether Currents

In a theory with Lagrangian density $\mathcal{L}(\Phi_i, \partial\Phi_i)$ (where the collective index i takes care of the different fields, as well as their spin type) the Noether current for space-time transformations has the form

$$J^\mu_f = \Pi^{\mu i}\delta_f\Phi_i + \mathcal{L}\delta_f x^\mu \ , \tag{7.1}$$

where $\delta_f x^\mu = f^\mu$, $\Pi^{\mu i} = \delta\mathcal{L}/\delta\partial_\mu\Phi_i$, $\delta_f\Phi_i = \Phi'_i(x) - \Phi_i(x)$, and the infinitesimal quantities f^μ take the following form

$$f^\mu = a^\mu \quad \text{or} \quad f^\mu = \omega^\mu_\nu x^\nu \quad \text{or} \quad f^\mu = a x^\mu \quad \text{or} \quad f^\mu = a^\mu x^2 - 2a\cdot x x^\mu \ , \tag{7.2}$$

for infinitesimal translations, rotations (and boosts), dilations, and special conformal transformations, respectively, where, as usual, $\omega^{\mu\nu} = -\omega^{\nu\mu}$.
With the current (7.1) one can: i) test whether the given transformation is a symmetry by picking the correspondent f^μ in (7.2), and checking whether $\partial_\mu J^\mu = 0$, by using the equations of motion; ii) use the Noether charges $Q_f \equiv \int d^3x J^0_f$, and the canonical equal-time Poisson brackets $\{\Phi_i(x), \Pi^j(y)\} = \delta^j_i \delta(\vec{x} - \vec{y})$, to generate the transformations of an arbitrary function of the canonical variables

$$\{G(\Phi_i, \Pi_i), Q_f\} \equiv \Delta_f G(\Phi_i, \Pi_i) \ . \tag{7.3}$$

Note also that, for $f^0 = g^0 = 0$,

$$\{Q_f, Q_g\} = Q_{[f,g]} \ , \tag{7.4}$$

and Eq.s (7.3) and (7.4) hold whether or not $\partial_0 Q_f = 0$. Of course, when Q_f acts on the fields it must reproduce the transformations one started with $\Delta_f \Phi_i = \delta_f \Phi_i$.

The expression for the current (7.1) has been obtained by varying the action, including the measure, under an arbitrary space-time transformation, and only afterwards one tests the invariance, following the above described procedure. Let us sketch here this proof based on Ref. [2].

Let us consider the Action

$$\mathcal{A}_\Omega = \int_\Omega d^4x \mathcal{L}(\Phi_i, \partial\Phi_i) \tag{7.5}$$

where Ω is the space-time volume of integration. The infinitesimal transformations of the coordinates, of the fields and of the derivatives of the fields are given respectively by

$$x_\mu \quad \rightarrow \quad x'_\mu = x_\mu + \delta_f x_\mu \tag{7.6}$$

$$\Phi_i(x) \quad \rightarrow \quad \Phi'_i(x') = \Phi_i(x) + \delta^*_f \Phi_i(x) \tag{7.7}$$

$$\partial_\mu \Phi_i(x) \quad \rightarrow \quad \partial'_\mu \Phi'_i(x') = \partial_\mu \Phi_i(x) + \delta^*_f \partial_\mu \Phi_i(x) \tag{7.8}$$

note that δ^*_f does not commute with the space-time derivatives.

When we act with this transformation the Action changes to

$$\mathcal{A}'_{\Omega'} = \int_{\Omega'} d^4x' \mathcal{L}(\Phi'_i, \partial'\Phi'_i)\,. \tag{7.9}$$

If the transformation is a symmetry we have $\mathcal{A}'_{\Omega'} - \mathcal{A}_\Omega = 0$, for *any field configuration*, i.e. off-shell, therefore at the first order we obtain

$$\begin{aligned} 0 &= \mathcal{A}'_{\Omega'} - \mathcal{A}_\Omega \\ &= \int_\Omega d^4x \Big[(1 + \partial_\rho \delta_f x^\rho)\Big(\mathcal{L}(\Phi_i, \partial\Phi_i) + \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta^*_f \Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta^*_f \partial_\mu \Phi_i\Big) - \mathcal{L}\Big] \\ &= \int_\Omega d^4x \Big(\frac{\partial \mathcal{L}}{\partial \Phi_i}\delta^*_f \Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta^*_f \partial_\mu \Phi_i + \mathcal{L}\partial_\rho x^\rho\Big) \end{aligned} \tag{7.10}$$

where $(1 + \partial_\rho \delta_f x^\rho)$ is the Jacobian of the change of coordinates from x' to x at the first order.

Let us now introduce another variation δ_f that commutes with the derivatives. If we do so we can write

$$\delta^*_f \Phi_i = \partial_\mu \Phi_i(x) \delta_f x^\mu + \delta_f \Phi_i \quad \text{and} \quad \delta^*_f(\partial_\mu \Phi_i) = \partial_\mu \partial_\nu \Phi_i(x) \delta_f x^\nu + \delta_f \partial_\mu \Phi_i\,. \tag{7.11}$$

Substituting these back in (7.10) we obtain

$$
\begin{aligned}
& \int_\Omega d^4x \Big[\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\delta_f\Phi_i\Big) + \Big(\frac{\partial\mathcal{L}}{\partial\Phi_i}\partial_\mu\Phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi_i)}\partial_\mu\partial_\nu\Phi_i\Big)\delta_f x^\mu + \mathcal{L}\partial_\mu\delta_f x^\mu\Big] \\
= & \int_\Omega d^4x \Big[\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\delta_f\Phi_i\Big) + \Big(\frac{\partial\mathcal{L}}{\partial x^\mu}\delta_f x^\mu + \mathcal{L}\partial_\mu\delta_f x^\mu\Big)\Big] \qquad (7.12) \\
= & \int_\Omega d^4x (\mathrm{E.L.})_i \delta_f\Phi^i \, ,
\end{aligned}
$$

which finally gives the wanted conservation law on-shell $\partial_\mu J^\mu = 0$ where

$$J^\mu = \Pi^\mu_i \delta_f \Phi^i + \mathcal{L}\delta_f x^\mu$$

If we write back the space-time dependent variations $\delta^*_f\Phi^i$ we obtain

$$
\begin{aligned}
J^\mu &= \Pi^\mu_i \delta^*_f\Phi^i - (\Pi^\mu_i\partial^\nu\Phi_i - \eta^{\mu\nu}\mathcal{L})\delta_f x_\nu \\
&= \Pi^\mu_i \delta^*_f\Phi^i - T^{\mu\nu}\delta_f x_\nu \qquad (7.13)
\end{aligned}
$$

that leads, for $f^\mu = a^\mu$, to the definition of the canonical (i.e. in general non-symmetric, non-traceless, etc.) energy-momentum tensor $T^{\mu\nu}$, and, for $f^\mu = \omega^{\mu\nu}x_\nu$, to the definition of the three rank tensor of angular momentum $M^{\mu\nu\lambda}$.

In Susy the Noether theorem has to be used in a somehow different perspective, namely by checking that $\delta\mathcal{L} = \partial_\mu V^\mu$, which is only true for invariant actions, and then writing the current as

$$J^\mu = N^\mu - V^\mu \qquad (7.14)$$

where, $N^\mu \equiv \Pi^{\mu i}\delta^*_f\Phi_i$ is the *rigid* current, and we recall that in this case $\delta^*_f\Phi_i = \Phi'_i(x') - \Phi_i(x)$.

The choice δ^*_f is the most useful in the case of gauge transformations and Susy, because in both cases it is not possible to write $V^\mu = -\mathcal{L}\delta_f x^\mu$. For gauge symmetry this is clearly due to the fact that they are internal symmetries, hence do not involve space-time transformations, and $\delta_f x^\mu = 0$.

For Susy, instead, the problem is more subtle. Here we list the specific problems for Susy-Noether currents:

First, Susy acts as the generator of translations only in superspace, hence its space-time nature is more involved.

Second, Susy transforms fermions into bosons and *vice versa* by transforming fermions into derivatives of the bosons and bosons into fermions. Therefore the conjugate momenta of the bosons appear in the Susy transformations of the fermions but the contrary is not true in general. This makes more difficult to express the full current in terms of canonical momenta and transformations.

Another point is that the linear realization of Susy involves bosonic Lagrange multipliers, the *dummy fields*, to which we cannot associate a conjugate momentum and the standard Noether procedure, based on such conjugates, breaks down. Of course the dummy fields can be eliminated by using their Euler-Lagrange equations but this introduces other ambiguities, unsolved in literature. Namely: *when* to put the dummy fields on-shell, before or after the computation of V_μ? Does that mean that all the fields have to be on-shell? Note that this last point is vital since the definition of symmetry in the first place is based on the variation off-shell of the Action.

Finally in SW theory we have to deal with effective Lagrangians and renormalization does not constraint the fermionic terms to be bilinear and the coefficients of the kinetic terms to be constant and in general this is not true. As a matter of fact, the SW effective Lagrangian is quartic in the fermionic fields and has coefficients of the kinetic terms that are non-polynomial functions of the scalar field. Because of this, the Noether procedure requires a great deal of care[1]. For example one encounters equal time commutations (Poisson brackets) between fermions and bosons such as

$$\{\psi \ , \ \pi_\phi\} = h(\phi)\psi \quad \text{from} \quad \{\pi_{\bar{\psi}} \ , \ \pi_\phi\} = 0 \tag{7.15}$$

where $h(\phi)$ is a non-polynomial function of the scalar field related to the coefficient of the kinetic terms. This reflects the difficulty of treating Noether currents in a quantum (effective) context.

All these problems can be solved by using the following recipe:

- The Susy-Noether charge that correctly reproduces the Susy transformations is the one obtained from $J^\mu = N^\mu - V^\mu$ where $\delta\mathcal{L} = \partial_\mu V^\mu$ and V^μ has to be extracted as it is, i.e. no terms like $\partial_\nu W^{[\nu\mu]}$ have to be added.

- The variation $\delta\mathcal{L}$ has to be performed off-shell by the definition of symmetry. Nevertheless the dummy fields, and *only* them, automatically are projected on-shell.

[1]Generally speaking, the Noether procedure has always to be handled with care when applied to quantum theories.

- The full current J^μ contains terms of the form $\pi_\psi \delta\psi$, that generate the fermionic transformations. The *same* term can be written as $\pi_\phi \delta\phi + \cdots$ therefore it also generates the bosonic transformations. The situation is more complicated for effective theories.

- The canonical commutation relations are preserved also at the effective level, even if some of the usual assumptions, such as that at equal time all fermions and bosons commute, are incorrect. Noether currents at the effective level do not exhibit the same simple expressions as at the classical level.

7.2 Computations of the SW Quantum Central Charge

As we hope is clear from the previous discussion, the mass formula

$$M = |Z| = \sqrt{2}|n_e a + n_m a_D| \tag{7.16}$$

plays a major role in SW model. Let us stress here again that the knowledge of the central charge Z amounts to the knowledge of the mass formula.
In a nutshell the important features of Z are:

- It allows for SSB of the gauge symmetry within the supersymmetric theory.

- It gives the complete and exact mass spectrum. Namely it fixes the masses for the elementary particles as well as the collective excitations.

- It exhibits an explicit SL(2, **Z**) duality symmetry whereas this symmetry is not a symmetry of the theory in the Noether sense.

- In the quantum theory it is the most important global piece of information at our disposal on $\mathcal{M}_q$. Therefore it is vital for the exact solution of the model.

It is then not surprising that, following the paper of Seiberg and Witten, there has been a big interest in the computation of the mass formula in the quantum case. As a matter of fact, in their paper there is no direct proof of this formula but only a check that the bosonic terms of the SU(2) high energy effective Hamiltonian for a magnetic monopole admit a BPS lower bound given by $\sqrt{2}|\mathcal{F}'(a)|$.
The complete and direct computation has to involve the Noether supercharges constructed from the Lagrangian.

7.2.1 U(1) Sector

Let us re-write here the U(1) low-energy lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{\mathrm{Im}}{4\pi}\int d^4x\Big(- \mathcal{F}''(\phi)[\partial_\mu\phi^\dagger\partial^\mu\phi + \frac{1}{4}F_{\mu\nu}\hat{F}^{\mu\nu} + i\psi\,\partial\!\!\!/\bar{\psi} + i\lambda\,\partial\!\!\!/\bar{\lambda} - (EE^\dagger + \frac{1}{2}D^2)] \\ &+ \mathcal{F}'''(\phi)[\frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}\psi F_{\mu\nu} - \frac{1}{2}(E^\dagger\psi^2 + E\lambda^2) + \frac{i}{\sqrt{2}}D\psi\lambda] + \mathcal{F}''''(\phi)[\frac{1}{4}\psi^2\lambda^2]\Big)\end{aligned}$$

This time the dummy fields couple non trivially to the fermions, while in the classical case $\mathcal{F} \to \frac{1}{2}\tau\phi^2$ they decouple from the dynamical fields. Their expression on-shell is given by

$$D = -\frac{1}{2\sqrt{2}}(f\psi\lambda + f^\dagger\bar{\psi}\bar{\lambda}) \tag{7.17}$$

$$E^\dagger = -\frac{i}{4}(f\lambda^2 - f^\dagger\bar{\psi}^2) \tag{7.18}$$

$$E = \frac{i}{4}(f^\dagger\bar{\lambda}^2 - f\psi^2) \tag{7.19}$$

where $f(\phi,\phi^\dagger) \equiv \mathcal{F}'''/\mathcal{I}$.
The non canonical momenta, namely the ones obtained before partial integration in the fermionic sector, are given by

$$\pi^\mu_\phi = -\mathcal{I}\partial^\mu\phi^\dagger \qquad \pi^\mu_{\phi^\dagger} = (\pi^\mu_\phi)^\dagger \tag{7.20}$$

$$\Pi^{\mu\nu} = -\frac{1}{2i}(\mathcal{F}''\hat{F}^{\mu\nu} - \mathcal{F}^{\dagger\prime\prime}\hat{F}^{\dagger\mu\nu}) + \frac{1}{i\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{\mu\nu}\psi - \mathcal{F}^{\dagger\prime\prime\prime}\bar{\lambda}\bar{\sigma}^{\mu\nu}\bar{\psi}) \tag{7.21}$$

and

$$(\pi^\mu_{\bar{\psi}})_{\dot{\alpha}} = \frac{1}{2}\mathcal{F}''\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^\mu_\psi)^\alpha = -\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \tag{7.22}$$

$$(\pi^\mu_{\bar{\lambda}})_{\dot{\alpha}} = \frac{1}{2}\mathcal{F}''\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^\mu_\lambda)^\alpha = -\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \tag{7.23}$$

where we scale $\mathcal{F}$ by a factor of 4π, and $\mathcal{F}'' = \mathcal{R} + i\mathcal{I}$.
The ϵ_1 computation gives

$$\begin{aligned}V^\mu_1 &= \delta_1\phi\pi^\mu_\phi + \frac{\mathcal{F}^{\dagger\prime\prime}}{\mathcal{F}''}\delta_1\bar{\psi}\pi^\mu_{\bar{\psi}} + \delta_1\psi\pi^\mu_\psi + \delta_1\lambda\pi^\mu_\lambda \\ &+\frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_\nu\bar{\lambda}F^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2\end{aligned} \tag{7.24}$$

This expression of V^μ_1 is far from being a straightforward generalization of the classical one. One could naively try to guess the effective V^μ_1 by simply "inverting the arrow" of the classical limit, $\mathcal{F}'' \leftarrow \tau$, but this is not the case.

Of course the rigid current is formally identical to the classical one, namely

$$N_1^\mu = \delta_1\phi\pi_\phi^\mu + \delta_1 A_\nu \Pi^{\mu\nu} + \delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu + \delta_1\psi\pi_\psi^\mu + \delta_1\lambda\pi_\lambda^\mu \tag{7.25}$$

Thus we can write down our total current as

$$\begin{aligned} J_1^\mu &\equiv N_1^\mu - V_1^\mu \\ &= \frac{2i\mathcal{I}}{\mathcal{F}''}\delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu + \Pi^{\mu\nu}\delta_1 A_\nu - \frac{1}{2i}\mathcal{F}^{\dagger''}\epsilon_1\sigma_\nu\bar{\lambda}F^{*\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 \end{aligned} \tag{7.26}$$

$$\begin{aligned} &= \sqrt{2}\mathcal{I}\epsilon_1(\partial\!\!\!/\phi^\dagger)\bar{\sigma}^\mu\psi - \frac{1}{2i}\mathcal{F}^{\dagger''}\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 \\ &\quad +[-\frac{1}{2}(\mathcal{F}''\hat{\phi}^{\mu\nu} - \mathcal{F}^{\dagger''}\hat{F}^{\dagger\mu\nu}) + \frac{1}{\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{\mu\nu}\psi - \mathcal{F}^{\dagger'''}\bar{\lambda}\bar{\sigma}^{\mu\nu}\bar{\psi})]\epsilon_1\sigma_\nu\bar{\lambda} \end{aligned} \tag{7.27}$$

Note that in the classical case, $\mathcal{F}'' \to \tau = \tau_R + i\tau_I$, this current becomes

$$J_1^\mu = \frac{2i\tau_I}{\tau}\delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu - \frac{2i\tau_I}{\tau^*}\delta_1^{\rm on}\lambda\pi_\lambda^\mu\,, \tag{7.28}$$

where $\delta_1\bar{\lambda} = 0$ and $\delta_1^{\rm on}\lambda$ stand for the variation of λ with dummy fields on-shell (there are no dummy fields in the variation of $\bar{\psi}$). In this case this means $E = D = 0$ and one could also wonder if they are simply cancelled in the total current. But, in agreement with our recipe, the dummy fields, and *only* them, have been automatically projected on-shell.
If we set $\theta = 0$, i.e. $\tau_R = 0$, in this non-canonical setting, we have

$$J_1^\mu|_{\theta=0} = 2(\delta_1^{\rm on}\lambda\pi_\lambda^\mu + \delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu) \tag{7.29}$$

and the same type of expression, $J^\mu = 2N^\mu_{\rm fermi}$, where N^μ is the rigid current, holds for the total current of the massive WZ model. We see that the double counting of the fermionic degrees of freedom provides a very compact formula for the currents. All the information is contained in the fermionic sector, since the variations of the fermions contain the bosonic momenta. Unfortunately this is not the case for the effective theory.
If we call $\mathcal{L}^I$ the Lagrangian with $\bar{\psi}$ and $\bar{\lambda}$ as fields, it differs from $\mathcal{L}$ only in the fermionic kinetic piece and $\mathcal{F}^{\dagger'''}$ type of terms

$$\mathcal{L}^I_{\rm kin.fermi} = \frac{1}{2i}[-\mathcal{F}''i\psi\ \partial\!\!\!/\bar{\psi} + \mathcal{F}^{\dagger''}i\psi\ \partial\!\!\!/\bar{\psi} - i\mathcal{F}^{\dagger'''}(\partial_\mu\phi^\dagger)\bar{\psi}\bar{\sigma}^\mu\psi + (\psi\to\lambda)] \tag{7.30}$$

Similarly for $\mathcal{L}^{II}$ (ψ and λ as fields)

$$\mathcal{L}^{II}_{\rm kin.fermi} = \frac{1}{2i}[-\mathcal{F}''i\bar{\psi}\ \bar{\partial\!\!\!/}\psi + \mathcal{F}^{\dagger''}i\bar{\psi}\ \bar{\partial\!\!\!/}\psi + i\mathcal{F}'''(\partial_\mu\phi)\psi\sigma^\mu\bar{\psi} + (\psi\to\lambda)] \tag{7.31}$$

The relation among the three Lagrangians is clearly

$$\mathcal{L} = \mathcal{L}^I + \partial_\mu(\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}(\bar\psi\bar\sigma^\mu\psi + \bar\lambda\bar\sigma^\mu\lambda)) \quad (7.32)$$

$$= \mathcal{L}^{II} + \partial_\mu(\frac{1}{2}\mathcal{F}''(\bar\psi\bar\sigma^\mu\psi + \bar\lambda\bar\sigma^\mu\lambda)) \quad (7.33)$$

and the momenta change accordingly.
Note that nothing changes for $\Pi^{\mu\nu}$, since

$$\Pi^{I\mu\nu} = \Pi^{II\mu\nu} = \Pi^{\mu\nu} \quad (7.34)$$

whereas in both cases $(\pi_\phi)^\dagger \neq \pi_{\phi^\dagger}$. From (7.32) follows that

$$V_1^{\mu I} = V_1^\mu - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\delta_1(\bar\psi\bar\sigma^\mu\psi + \bar\lambda\bar\sigma^\mu\lambda) \quad (7.35)$$

where $\delta_1\mathcal{L}^I = \partial_\mu V_1^{\mu I}$ and $\delta_1\mathcal{F}^\dagger = 0$. Explicitly (7.35) reads

$$\begin{aligned} V_1^{\mu I} &= \delta_1\phi\pi^\mu_\phi + \frac{\mathcal{F}''^\dagger}{\mathcal{F}''}\delta_1\bar\psi\pi^\mu_{\bar\psi} + \delta_1\psi\pi^\mu_\psi + \delta_1\lambda\pi^\mu_\lambda \\ & + \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar\lambda F^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar\psi\bar\lambda^2 \\ & - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\delta_1\bar\psi\bar\sigma^\mu\psi - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar\psi\bar\sigma^\mu\delta_1\psi - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar\lambda\bar\sigma^\mu\delta_1\lambda \end{aligned} \quad (7.36)$$

the second, third and fourth terms in the first line cancel against the first, second and third terms in the third line respectively, therefore the fermionic momenta are absent from $V_1^{\mu I}$. But also the rigid current changes to

$$N_1^{\mu I} = \delta_1\phi\pi^\mu_\phi + \Pi^{\mu\nu}\delta_1 A_\nu + \delta_1\bar\psi\pi^{I\mu}_{\bar\psi} \quad (7.37)$$

thus, recalling that $J_1^{\mu I} = N_1^{\mu I} - V_1^{\mu I}$, we have

$$J_1^{\mu I} = \delta_1\bar\psi\pi^{I\mu}_{\bar\psi} + \Pi^{\mu\nu}\delta_1 A_\nu - \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar\lambda F^{*\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar\psi\bar\lambda^2 \quad (7.38)$$

which is identical to the one in (7.27) when we write explicitly the transformations and the *new* momenta.
If we choose the current (7.38), the charge is given by

$$\epsilon_1 Q_1^I = \int d^3x(\delta_1\bar\psi\pi^I_{\bar\psi} + \Pi^i\delta_1 v_i - \frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_i\bar\lambda F^{*0i} - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}\epsilon_1\sigma^0\bar\psi\bar\lambda^2) \quad (7.39)$$

Let us first write the R-symmetric ($1 \leftrightarrow 2$: $\psi \leftrightarrow -\lambda$, $E \leftrightarrow E^\dagger$, and $F_{\mu\nu} \leftrightarrow -F_{\mu\nu}$) of (7.39) given by

$$\epsilon_2 Q_2^I = \int d^3x \Big(\delta_2 \bar{\lambda} \pi_{\bar{\lambda}}^I + \Pi^j \delta_2 A_j - \frac{1}{2i} \mathcal{F}^{\dagger''} \epsilon_2 \sigma_j \bar{\psi} F^{*0j} + \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger'''} \epsilon_2 \sigma^0 \bar{\lambda} \bar{\psi}^2 \Big) \tag{7.40}$$

Now we have to commute the two charges paying due attention to the subtleties of the *effective* Poisson brackets:

$$\begin{aligned} \{\epsilon_1 Q_1, \epsilon_2 Q_2\}_- \quad = \quad & \int d^3x \Big(\partial_i [\epsilon_1 \epsilon_2 (2\sqrt{2} \Pi^i \phi^\dagger + \sqrt{2} F^{*0i} \mathcal{F}'^\dagger) + i \mathcal{F}^{\dagger''} (\epsilon_1 \sigma^0 \bar{\psi} \epsilon_2 \sigma^i \bar{\lambda} - \epsilon_2 \sigma^0 \bar{\lambda} \epsilon_1 \sigma^i \bar{\psi})] \\ & + [2\sqrt{2} (\partial_i \Pi^i) \phi^\dagger + \sqrt{2} (\partial_i F^{*0i}) \mathcal{F}'^\dagger] \epsilon_1 \epsilon_2 \Big) \end{aligned} \tag{7.41}$$

Imposing the Bianchi identities and the Gauss law, dropping the Susy parameters ϵ_1 and ϵ_2, using the formula $Z = \frac{i}{4} \epsilon^{\alpha\beta} \{Q_{1\alpha}, Q_{2\beta}\}_+$, dropping the fermionic terms, and reintroducing the factor 4π we can write

$$Z = i\sqrt{2} \int d^2\vec{\Sigma} \cdot (\vec{\Pi} \phi^\dagger + \frac{1}{4\pi} \vec{B} \phi_D^\dagger) \tag{7.42}$$

where $d^2\vec{\Sigma}$ is the measure on the sphere at infinity S^2_∞, $B^i = \frac{1}{2} \epsilon^{0ijk} F_{jk}$ and the SW dual of the scalar field $\phi^\dagger$ is then computed to be

$$\phi_D^\dagger \equiv \mathcal{F}'^\dagger (\phi^\dagger) \tag{7.43}$$

Surprisingly enough the expression (7.42) is *formally* identical to the classical one. We see that the topological nature of Z is strong enough to protect its form at the quantum level. All one has to do is to use a little dictionary and replace classical quantities by their quantum counterparts.
Thus we can apply exactly the same logic as in the classical case and define the electric and magnetic charges *à la* Witten and Olive. The final expression is

$$Z = i\sqrt{2} (n_e a^* + n_m a_D^*) \tag{7.44}$$

where $< 0|\phi^\dagger|0 >= a^*$, $< 0|\phi_D^\dagger|0 >= a_D^*$ and n_e, n_m are the electric and magnetic quantum numbers, respectively.
Eventually we proved the SW mass formula. At this end we can simply use the BPS type of argument noticing that our direct computation includes fermions but they occur as a total divergence which falls off fast enough to give contribution on S^2_∞. Thus

$$M = |Z| = \sqrt{2} |n_e a + n_m \mathcal{F}'(a)| \tag{7.45}$$

A last remark is now in order. The U(1) low energy theory is invariant under the linear shift $\mathcal{F}(\phi) \to \mathcal{F}(\phi) + c\phi$. This produces an ambiguity in the definition of Z. For this and other purposes we want also to analyze the SU(2) high energy theory.

7.2.2 SU(2) Sector

Fortunately we do not have to repeat the computation of V_μ in this sector since we can generalize the U(1) result to the SU(2) one: $Q_1^{\mathrm{U}(1)} \to Q_1^{\mathrm{SU}(2)}$. The way to do that is to take the simplest SU(2) generalization of the U(1) charge and check whether it reproduces the right SU(2) Susy transformations. This is not actually the case, and a term has to be added. This term is unique. The final result of this procedure is

$$\begin{aligned}\epsilon_1 Q_1 &= \int d^3x \Big(\Pi^{ai}\delta_1 A_i^a + \delta_1\bar{\psi}^a \pi^a_{\bar{\psi}} + \frac{i}{2}\mathcal{F}^{\dagger ab}\epsilon_1\sigma_i\bar{\lambda}^a F^{*0ib} \\ &\quad - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger abc}\epsilon_1\sigma^0\bar{\psi}^a\bar{\lambda}^b\bar{\lambda}^c + i\mathcal{I}^{ab}\epsilon_1\sigma^0\bar{\lambda}^b\epsilon^{acd}\phi^c\phi^{d\dagger}\Big) \qquad (7.46)\end{aligned}$$

where some care is necessary in handling the derivatives of the prepotential $\mathcal{F}(\phi^a\phi^a)$, function of the SU(2) Casimir $\phi^a\phi^a$. The first four derivatives are given by

$$\mathcal{F}^a = 2\phi^a\mathcal{F}' \qquad (7.47)$$

$$\mathcal{F}^{ab} = 2\delta^{ab}\mathcal{F}' + 4\phi^a\phi^b\mathcal{F}'' \qquad (7.48)$$

$$\mathcal{F}^{abc} = 4(\delta^{ab}\phi^c + \delta^{ac}\phi^b + \delta^{bc}\phi^a)\mathcal{F}'' + 8\phi^a\phi^b\phi^c\mathcal{F}''' \qquad (7.49)$$

$$\begin{aligned}\mathcal{F}^{abcd} &= 4\mathcal{F}''(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}) + 8\mathcal{F}'''(\phi^a\phi^b\delta^{cd} + \phi^a\phi^c\delta^{bd} + \phi^a\phi^d\delta^{bc} \\ &\quad + \phi^b\phi^c\delta^{ad} + \phi^b\phi^d\delta^{ac} + \phi^c\phi^d\delta^{ab}) + 16\mathcal{F}''''\phi^a\phi^b\phi^c\phi^d \qquad (7.50)\end{aligned}$$

similarly for $\mathcal{F}^\dagger$.
The R-symmetric counterpart of (7.46) is given by

$$\begin{aligned}\epsilon_2 Q_2 &= \int d^3x \Big(\Pi^{bj}\delta_2 A_j^b + \delta_2\bar{\lambda}^b \pi^b_{\bar{\lambda}} + \frac{i}{2}\mathcal{F}^{\dagger cd}\epsilon_2\sigma_j\bar{\psi}^c F^{*0jd} \\ &\quad + \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger def}\epsilon_2\sigma^0\bar{\lambda}^d\bar{\psi}^e\bar{\psi}^f + i\mathcal{I}^{ef}\epsilon_2\sigma^0\bar{\psi}^f\epsilon^{egh}\phi^g\phi^{h\dagger}\Big) \qquad (7.51)\end{aligned}$$

The complex conjugate of (7.46) is given by

$$\begin{aligned}\bar{\epsilon}_1\bar{Q}_1 &= \int d^3x \Big(\Pi^{ai}\bar{\delta}_1 A_i^a + \sqrt{2}\mathcal{I}^{ab}\bar{\epsilon}_1 \,\bar{\mathcal{D}}\phi^a\sigma^0\bar{\psi}^b + \frac{i}{2}\mathcal{F}^{ab}\bar{\epsilon}_1\sigma_i\lambda^a F^{*0ib} \\ &\quad + \frac{1}{2\sqrt{2}}\mathcal{F}^{abc}\bar{\epsilon}_1\bar{\sigma}^0\psi^a\lambda^b\lambda^c - \bar{\epsilon}_1\pi^a_{\bar{\lambda}}\epsilon^{acd}\phi^c\phi^{d\dagger}\Big) \qquad (7.52)\end{aligned}$$

where we introduced the conjugate momentum $\pi^a_{\bar{\lambda}}$.
The final formula for H is

$$H = -\frac{i}{4}\bar{\sigma}^{0\dot{\alpha}\alpha}\{Q_{1\alpha}\,,\ \bar{Q}_{1\dot{\alpha}}\}_+ \qquad (7.53)$$

where we defined $H = P^0 = -P_0$. This lengthy computation gives

$$\begin{aligned}
H \quad = \quad & \int d^3x\Big(-\frac{1}{2}(\mathcal{I}^{ab})^{-1}\Pi^{ai}\Pi^{bi} - (\mathcal{I}^{ab})^{-1}\mathcal{R}^{bc}\Pi^{ai}B^{ic} - \frac{1}{2}(\mathcal{I}^{ab})^{-1}\mathcal{F}^{\dagger ab}\mathcal{F}^{ef}B^{ib}B^{if} \\
& -(\mathcal{I}^{ab})^{-1}\pi^a_\phi(\pi^b_\phi)^\dagger + \mathcal{I}^{ab}(\mathcal{D}^i\phi^a)(\mathcal{D}^i\phi^{\dagger b}) \\
& +i\mathcal{I}^{ab}\psi^a\sigma^i\mathcal{D}_i\bar{\psi}^b + i\mathcal{I}^{ab}\lambda^a\sigma^i\mathcal{D}_i\bar{\lambda}^b + \frac{1}{2}(\partial_i\mathcal{F}^{\dagger ab})\bar{\lambda}^a\sigma^i\lambda^b \\
& -\frac{1}{\sqrt{2}}\mathcal{I}^{ad}\epsilon^{abc}(\phi^c\bar{\psi}^d\bar{\lambda}^b + \phi^{\dagger c}\psi^d\lambda^b) + \frac{1}{2}\mathcal{I}^{ab}\epsilon^{acd}\epsilon^{bfg}\phi^c\phi^{\dagger d}\phi^f\phi^{\dagger g} \\
& +\frac{i}{\sqrt{2}}(\mathcal{I}^{af})^{-1}\mathcal{F}^{feg}\psi^e\sigma_{i0}\lambda^g(\Pi^{ia} + \mathcal{F}^{\dagger ab}B^{ib}) \\
& -\frac{i}{\sqrt{2}}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{\dagger abc}\bar{\psi}^a\bar{\sigma}_{i0}\bar{\lambda}^b(\Pi^{ie} + \mathcal{F}^{ed}B^{id}) \\
& +\frac{1}{16}\mathcal{F}^{\dagger efg}\mathcal{F}^{cad}(\mathcal{I}^{gc})^{-1}\bar{\psi}^e\bar{\psi}^f\psi^a\psi^c + \frac{1}{16}\mathcal{F}^{\dagger abc}\mathcal{F}^{efg}(\mathcal{I}^{ae})^{-1}\bar{\lambda}^b\bar{\lambda}^c\lambda^f\lambda^g \\
& +\frac{3}{16}\mathcal{F}^{\dagger bec}\mathcal{F}^{\dagger efg}(\mathcal{I}^{ab})^{-1}\bar{\psi}^a\bar{\psi}^c\bar{\lambda}^f\bar{\lambda}^g + \frac{3}{16}\mathcal{F}^{bec}\mathcal{F}^{efg}(\mathcal{I}^{ab})^{-1}\psi^a\psi^c\lambda^f\lambda^g \\
& -\frac{1}{2i}(\frac{1}{4}\mathcal{F}^{abcd}\psi^a\psi^b\lambda^c\lambda^d - \frac{1}{4}\mathcal{F}^{\dagger abcd}\bar{\psi}^a\bar{\psi}^b\bar{\lambda}^c\bar{\lambda}^d)\Big) \\
& +\int d^3x\partial_i(\frac{1}{2}\mathcal{F}^{\dagger ab}\bar{\lambda}^a\bar{\sigma}^i\lambda^b - \frac{i}{2}\mathcal{I}^{ab}\bar{\psi}^a\bar{\sigma}^i\psi^b) \qquad (7.54)
\end{aligned}$$

where $E^{ai} = F^{a0i}$ and $B^{ai} = \frac{1}{2}\epsilon^{0ijk}F^a_{jk}$ are the SU(2) generalization of the electric and magnetic fields, respectively.
The Gauss law is obtained by

$$\begin{aligned}
0 = \frac{\partial\mathcal{L}}{\partial A^{g0}} \quad = \quad & \frac{1}{2i}[\partial_i(-\mathcal{F}^{gb}\hat{F}^{bi}_0 + \sqrt{2}\mathcal{F}^{gbc}\lambda^b\sigma^i_0\psi^c) \\
& -\epsilon^{gad}A^{id}\mathcal{F}^{ab}\hat{F}^b_{i0} + \epsilon^{gcd}\sqrt{2}\mathcal{F}^{abc}\lambda^a\sigma^i_0\psi^bA^d_i - h.c.] \\
+ \quad & \epsilon^{gac}\mathcal{I}^{ab}(\phi^c\mathcal{D}_0\phi^{\dagger b}\phi^{\dagger c}\mathcal{D}_0\phi^b + i\psi^b\sigma_0\bar{\psi}^c + i\lambda^b\sigma_0\bar{\lambda}^c) \quad (7.55)
\end{aligned}$$

Recalling the definition of the conjugate momentum Π^{gi} of A^g_i and the definition of the covariant derivative, $\mathcal{D}_\mu X^a = \partial_\mu X^a + \epsilon^{abc}A^b_\mu X^c$, we see that the first two lines give $\mathcal{D}_i\Pi^{gi}$. Thus we have

$$\mathcal{D}_i\Pi^{ig} = -\epsilon^{gac}\mathcal{I}^{ab}(\phi^c\mathcal{D}^0\phi^{\dagger b} + \phi^{\dagger c}\mathcal{D}^0\phi^b + i\psi^b\sigma^0\bar{\psi}^c + i\lambda^b\sigma^0\bar{\lambda}^c) \qquad (7.56)$$

which is the required Gauss law.
The central charge is given by

$$\begin{aligned}
Z \quad = \quad & \int d^3x\Big(\partial_i[i\sqrt{2}(\Pi^{ai}\phi^{\dagger a} + B^{ai}\phi^{\dagger a}_D) - \mathcal{F}^{\dagger ab}\bar{\psi}^a\bar{\sigma}^{i0}\bar{\lambda}^b] \\
+ \quad & i\sqrt{2}[(\mathcal{D}_i\Pi^{ai})\phi^{\dagger a} + i\mathcal{I}^{be}\epsilon^{bcd}\phi^{\dagger d}(\psi^e\sigma^0\bar{\psi}^c + \lambda^e\sigma^0\bar{\lambda}^c) \\
& -\epsilon^{abc}\phi^b\phi^{\dagger c}\pi^a_\phi]\Big) \qquad (7.57)
\end{aligned}$$

We see from here that the terms which are not a total divergence, given in the second and third lines above, simply cancel due to the Gauss law (7.56). Eventually we are left with the surface terms that vanish when the SU(2) gauge symmetry is not broken down to U(1). If we break the symmetry along a flat direction of the Higgs potential, say $a = 3$, we recover the same result we found in the U(1) sector. In other words we see that on the sphere at infinity

$$\begin{aligned} Z &= \int d^2\vec{\Sigma} \cdot [i\sqrt{2}(\vec{\Pi}^a \phi^{\dagger a} + \frac{1}{4\pi}\vec{B}^a \phi_D^{\dagger a}) - \frac{1}{4\pi}\mathcal{F}^{\dagger ab}\bar{\psi}^a \vec{\bar{\sigma}} \bar{\lambda}^b)] \\ &\to i\sqrt{2}\int d^2\vec{\Sigma} \cdot (\vec{\Pi}^3 \phi^{\dagger 3} + \frac{1}{4\pi}\vec{B}^3 \phi_D^{\dagger 3}) \end{aligned} \qquad (7.58)$$

where $\vec{\bar{\sigma}} \equiv (\bar{\sigma}^{01}, \bar{\sigma}^{02}, \bar{\sigma}^{03})$ and we reintroduced the factor 4π. We made the usual assumption that the bosonic massive fields in the SU(2)/U(1) sector ($a = 1, 2$) and all the fermionic fields fall off faster than r^3, whereas the scalar massless field ($a = 3$) and its dual tend to their Higgs v.e.v.'s a^* and a_D^*, respectively.
We conclude that the fields in the massive sector, have no effect on the mass formula.

7.3 Exercises

Exercise VII.a For any space time symmetry the Noether current has to be written as $J^\mu = N^\mu - V^\mu$, as explained in details for SUSY. What is V^μ for

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

for space-time translations? What is the effect of the freedom of redefining V^μ ($V^\mu \to V'^\mu = V^\mu + \partial_\nu W^{\mu\nu}$ where $W^{\mu\nu} = -W^{\nu\mu}$) on the current you obtained? In the presence of gravity this effect has some physical consequences, which ones?
Exercise VII.b Explain why, even if the fermionic fields ψ and λ are present in the central charge Z, they do not contribute to the mass formula. (Hint: this happens in the classical as well as quantum case, in the U(1) and SU(2) sectors of the theory).

7.4 Further Reading

[1] A.Iorio, L.O'Raifeartaigh, S.Wolf, Ann. Phys. 290 (2001) 156; [2] J.Lopuszanski, "An introduction to Symmetry and Supersymmetry in quantum field theory", World Scientific, 1991.

Chapter 8

Short Talks

20 minutes for the talk + 5 minutes for questions/discussion.

PRAGUE - 2 May 2002
Speakers:

Tomaš Brauner Coset construction of superspace and super-covariant derivatives. Refs. [1] Callan, S.Coleman, J.Wess, B.Zumino, Phys. Rev. 177 (1969) 2239 (this paper, of course, does not refer to Susy, but to effective theories...); [2] J.Bagger, hep-ph/9604232 (this refers to Susy)

Michal Malinský Absence of quadratic divergencies in the interacting WZ model. Ref. [1] J.Wess, B.Zumino, Phys. Lett. B 49 (1974) 52.

Tomaš Sýkora SSusyB (LOR model). Ref. [1] D.Bailin and A.Love, "Supersymmetric gauge field theory and string theory" Chp 2.

SALERNO - 8 July 2002
Speakers:

Fabrizio Canfora (Anti-)Self Dual Gravity.

Filippo Maimone HD Gravity and Pauli-Villars regularization.

Stefano Sellitto Searching for Experimental Signatures of Susy at LHC.

Alessio Serafini Fiber Bundles and Dirac Monopoles.

Antonio Troisi Introduction to Sugra.

Bibliography

[1] J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton University Press, 1992.

[2] P. West, Introduction to Supersymmetry and Supergravity, World Scientific, 1986.

[3] S. Weinberg, The Quantum Theory of Fields, Vol. III Supersymmetry, Cambridge University Press, 2000.

[4] C. Gomez and R. Hernandez, Electric-Magnetic Duality and Effective Field Theory, hep-th/ 9510023.

[5] S. Coleman, Aspects of Symmetry, Cambridge University Press, 1985.

[6] A. Iorio, Supersymmetric Noether Currents and Seiberg-Witten Theory, PhD Thesis, Trinity College Dublin 1999 - [hep-th/0006198].