Determination of the light quark masses
The $m_u$ and $m_d$ quark masses from $\eta \rightarrow 3\pi$ decay

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Work performed in cooperation with K. Kampf, M. Knecht and J. Novotný
[aka Prague-(Lund)-Marseille Dispersive Treatment].
Determination of the current masses of $u$, $d$ and $s$ quarks

- quark confinement prohibits their direct determination
- determination – from comparison of theoret. prediction for some observable depending on quark masses with the corresponding experimental value

Methods

- effective theories for QCD – chiral perturbation theory
- QCD sum-rules – analytic properties of hadronic spectral functions + OPE
- lattice QCD

Advantages and disadvantages of SR and lattice

- $m_s$ and $\hat{m} = \frac{m_u + m_d}{2}$ determined independently with compatible results
- isospin breaking effects on the observables studied by these methods generated by $m_u - m_d$ difference and by EM interactions of the same order, however, elmag. corrections problematic for both of them
- isospin breaking effects in both methods nowadays need additional input
Use of $\chi$PT

- can determine only quark mass ratios and needs some input fixing the physical definition of the masses (Kaplan-Manohar ambiguity)
- For a more precise determination of the light quark masses it is therefore useful to combine isospin symmetric results of lattice QCD and sum rules with some isospin breaking study performed in $\chi$PT.

$\eta \to 3\pi$ in $\chi$PT

- proceeds via IB effects; moreover direct effect of EM on the amplitude small $\Rightarrow$ proportional to $m_d - m_u \sim \frac{1}{R}$; measurement of its decay rate $\sim R$
- two-loop $\chi$PT computation exists, but
  - large corrections in first three successive orders
  - discrepancies between the experimentally measured and $\chi$PT predicted Dalitz parameters (describing energy dependence)
  - poor knowledge of $C_i$
- it inspired studies of the origin of the discrepancy and its effect on $R$
η → 3π: Alternative approaches

Origin of the discrepancy?

- incorrect determination of NNLO LECs $C_i$, effect of resonances
- higher-order final state rescatterings
- influence of slow convergence of $\pi\pi$ scattering or $\eta \rightarrow 3\pi$ amplitude
- unexpectedly large electromagnetic contribution, . . .

[B. Kubis, S. Schneider, C. Ditsche; S. Lanz, G. Colangelo, E. Passemar; M. Kolesár, J. Novotný; A. Nehme, S. Zein.]

Its influence on $R$ determination

The approaches taking different assumptions than ChPT do not fix normalization (there do not appear $m_q$ explicitly).

⇒ in order to address this question, unavoidable to match to ChPT
    ⇒ need to find a region where both these approaches compatible.

The only approaches addressing this question by employing the $\eta \rightarrow \pi^+\pi^-\pi^0$ so far are the two dispersive approaches.
Dispersive approaches – the main differences

- Isospin breaking parameters: 
  \( \hat{m} = \frac{m_u + m_d}{2}, \quad r = \frac{m_s}{\hat{m}} \)

\[
Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}
\]

\[
\leftarrow Q = \frac{1}{2} R(r + 1) \rightarrow
\]

\[
R = \frac{m_s - \hat{m}}{m_d - m_u}
\]

- At NLO

**Advantages of using \( Q \)**
- Expressible using only QCD meson masses
- Reasonably stable w.r.t. Kaplan-Manohar

**Advantages of using \( R \)**
- Nothing special (maybe better connection to baryon physics?)
Dispersive approaches – the main differences

- isospin breaking parameters ($\hat{m} = \frac{m_u + m_d}{2}$, $r = \frac{m_s}{\hat{m}}$)

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Advantages of using $R$
- Nothing special (maybe better connection to baryon physics?)

- at higher orders

Advantages of using $Q$
- both of them lost – the relation obtains $r$-dependent corrections
- better connection to NLO
- historical reasons

Advantages of using $R$
- Direct connection to the two-loop $\chi$PT computation - not explicitly dependent on $r$ (but determination of values of $L_i$ is)
- Kaplan-Manohar has to be fixed from elsewhere ($L_i$s from Lattice, large $N_c$, . . .)
Dispersive approaches – the main differences

What do we have in common?

- construct a parametrization by using dispersive relations and the two-particle unitarity ⇒ dispersive
- parameters stem from the subtraction polynomials of the dispersive relations
- use that parametrization for correcting the chiral results or more-or-less fit the experimental data in order to determine $R$

The most obvious differences:

**Analytical perturbative dispersive approach of KKNZ**

- proceeds order by order in the construction of the amplitude
- the original guiding principle was a direct correspondence to $\chi$ amplitude
- all of the computations are performed analytically

**Numerical fully dispersive approach of Bern**

- searches a stable point of the dispersive relations
- the original guiding principle was to include $\pi\pi$ rescatterings to all orders
- the computations are performed numerically
Dispersive approaches – the main differences

The form of the parametrization

\[ M(s, t, u) = \text{Normalization} \ (\text{Polynomial} + \text{Unitary part}) \]

Normalization:

- **KKNZ**: Constant
- **Bern**: Omnès function containing elastic \( \pi \pi \) rescattering in S-channel

Polynomial:

- 6 parameters – a reasonable way how to estimate the error of neglecting higher imaginary parts of the parameters – can be added or neglected (good estimate)
- contain many various contributions that cannot be separated
- with a good statistics of data a clean fit (up to the normalization) possible

- 6 parameters – less number of parameters was insufficient
- imaginary parts of the parameters – Taylor expand the amplitude and seems its coefficients small or can be added
- should be more stable w.r.t. iterations; effect of \( \pi \pi \) rescattering separated
- do they use the Adler staff really just for fixing the normalization?
Dispersive approaches – the main differences

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Unitary part:
- stems from the $\pi\pi$ rescattering in all (crossed) channels
- parameters from Polynomial multiplied by polynomials in Mandelstam variables and by combinations of 5 kinematical functions
- depends on $\pi\pi$ scattering parameters
- inclusion of higher than 2-loop orders only through addition of higher chiral parts of constants and physical $\pi\pi$ p.
Dispersive approaches – the main differences

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Additional extensions:
Would you like to add inelastic contributions for the price of having more free parameters? O.K.
Would you like to have results that take into account \( m_{\pi \pm} \neq m_{\pi^0} \)? O.K. with additional free parameters.
Fits to our parametrization

Fit of a model

The amplitude can be written in the form

\[ M(s, t, u) = A_x f_A(s, t, u) + B_x f_B(s, t, u) + C_x f_C(s, t, u) + D_x f_D(s, t, u) 
+ E_x (s - s_0)^3 + F \left[ (t - s_0)^3 + (u - s_0)^3 \right], \]

where \( f_i(s, t, u) \) are "simple" functions of \( s, t, u \) and \( \pi\pi \) scattering parameters.

⇒ simple linear fit of the model

Fit of NNLO ChPT

Worked well.
We have tried to add also higher order \( \pi\pi \) rescattering by changing the unitarity part → the Dalitz plot parameters moved in the right direction, but the difference was reduced just by 10%. We would need to change also the polynomial part.
Fit of a model

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Fit of an experiment

Data are on \(|M(s, t, u)|^2\), which is not linear but quadratic in the parameters. But it has worked so far for:

- very optimistic data set constructed from KLOE 2008 (2500 data points)
- realistic data set constructed from KLOE 2008 (154 points)

Analysis of real data from WasaCOSY 20 with 50 data points under progress.
Our analysis I: correcting values of $C_i$'s

Even though the individual Dalitz parameters differ significantly, there exist $C_i$-independent combinations of them and their central values are in concordance. ⇒ We study this possibility.

<table>
<thead>
<tr>
<th>$C_i$-independent combinations of Dalitz parameters ($m_{\pi\pm} = m_{\pi^0}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. rel$_1$ : $4(b + d) - a^2 - 16\alpha$</td>
</tr>
<tr>
<td>2. rel$_2$ : $a^3 - 4ab + 4ad + 8f - 8g$</td>
</tr>
<tr>
<td>3. rel$_3$ : $\beta$</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Dalitz Parameter</th>
<th>KLOE</th>
<th>$\chi$PT</th>
</tr>
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<tbody>
<tr>
<td>$a$</td>
<td>$-1.09 \pm 0.02$</td>
<td>$-1.271 \pm 0.075$</td>
</tr>
<tr>
<td>$b$</td>
<td>$0.124 \pm 0.012$</td>
<td>$0.394 \pm 0.102$</td>
</tr>
<tr>
<td>$d$</td>
<td>$0.057 \pm 0.017$</td>
<td>$0.055 \pm 0.057$</td>
</tr>
<tr>
<td>$f$</td>
<td>$0.14 \pm 0.02$</td>
<td>$0.025 \pm 0.160$</td>
</tr>
<tr>
<td>$g$</td>
<td>$\sim 0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-0.030 \pm 0.005$</td>
<td>$0.013 \pm 0.016$</td>
</tr>
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</table>

Change of $C_i$'s reflects in a shift of the polynomial part of the amplitude $\Delta P(s, t, u)$. We fit the "corrected amplitude" to KLOE parametrization. ⇒ Reasonably small $\Delta P$.

Optimistic KLOE distribution

\[ R = 37.7 \pm 2.9. \]

More conservative estimate

\[ R = 30^{+14}_{-12}. \]
imposing nothing about the origin of the Dalitz paramet. discrepancy

the parametrization employs just general properties of $M(s, t, u)$ and the observed hierarchy of various contributions ⇒ Whatever the explanation of the discrepancies is, we assume the physical amplitude fulfills these assumptions.

Physical normalization of the amplitude

We need at least 1 point from ChPT, where physics is reproduced well.
Where is the Adler zero?

- in the \( m_u = m_d = 0 \) limit the amplitude in the point \( s = t = 0 \) (and its \( s = u \) brother) has to be zero
- this point can move anywhere in the distance of order \( O(m^2_\pi) \)
- at that point the (real and imaginary parts of) amplitude is of order \( O(m^2_\pi) \), i.e. if we use the naturality condition, it should be there significantly smaller than say at the center of the Dalitz plot
- no statement about chiral convergence of the amplitude
Our requirements for the correct regions

- the matching should be dependent on the values of $C_i$'s as less as possible; ✓
- within this region the chiral expansion should work well;
- also the higher corrections should be small there;
- the physical amplitude should have the similar behavior as the chiral amplitude inside the region.

The exact procedure for fixing the normalization

- match only the imaginary part of the ChPT under the physical threshold
Physical normalization of the amplitude – results

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Result of this analysis: $R = 39.6^{+2.5}_{-5.1}$
Lesson in the extrapolations

Real part of the amplitude in the \( s = u \) cut
Lesson in the extrapolations

Real part of the amplitude in the $s = u$ cut
Lesson in the extrapolations

Real part of the amplitude in the \( s = u \) cut

Statistical error in \( F \)
Lesson in the extrapolations

Real part of the amplitude in the $s = u$ cut

Mild dependence on $F$
Lesson in the extrapolations

Real part of the amplitude in the \( s = u \) cut

![Graph of the real part of the amplitude in the \( s = u \) cut](image1)

Imaginary part in the \( s = u \) cut

![Graph of the imaginary part of the amplitude in the \( s = u \) cut](image2)
Combined result

These two analyses have different assumptions and sources of errors, however, lead to compatible results, we can combine them into

**Final result for OPT distribution**

\[
R = 37.7 \pm 2.2.
\]

Estimating the error connected with the dependence on the “physical” distribution used in the fit, the more conservative value is

**Final result**

\[
R_2 = 39.6^{+2.5}_{-5.1}.
\]

Combination with the other constraints on quark masses from SR, Lattice, ...

Our final results for \(m_q\) characteristics

\[
\frac{m_u}{m_d} = 0.50^{+0.02}_{-0.05} \ [0.48(2)],
\]

\[
Q = 23.7^{+0.8}_{-1.5} \ [23.2(7)],
\]

quark masses in \(\overline{MS}\) at \(\mu = 2\) GeV

\[
m_u = 2.29^{+0.10}_{-0.17} \text{ MeV} \ [2.21(9) \text{ MeV}],
\]

\[
m_d = 4.57^{+0.19}_{-0.14} \text{ MeV} \ [4.62(12) \text{ MeV}].
\]