Quantum Phase Transitions in Nuclear Collective Models
(Links to Other Concepts)

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NIL DESPERANDUM!
(every suffering has end (when new suffering starts (but through suffering we grow)))
“Now, Nina, do you think you could throw something into the sea?”
“I think I could,” replied the child, “but I am sure that Pablo would throw it a great deal further than I can.”
“Never mind, you shall try first.”
Putting a fragment of ice into Nina’s hand, he addressed himself to Pablo: “Look out, Pablo; you shall see what a nice little fairy Nina is! Throw, Nina, throw, as hard as you can.”

Nina balanced the piece of ice two or three times in her hand, and threw it forward with all her strength.

A sudden thrill seemed to vibrate across the motionless waters to the distant horizon, and the Gallian Sea had become a solid sheet of ice!
Quantum phase transitions (QPTs)
phase transitions at zero temperature driven by interaction strengths

Lattice systems:

Many-body systems: mostly atomic nuclei
- DJ Thouless (1961): “collapse of RPA”
- R Gilmore (late 1970’s): “ground state (energy) phase transition”
  Lipkin model (toy many-body model formulated in terms of quasispin operators)
  Dicke model (laser phase transition in an interacting atom-field system)
- AEL Dieperink, O Scholten, F Iachello (1980):
  Interacting boson model (nuclear shape transitions)
- WM Zhang, DH Feng, JN Ginocchio (1987):
  Fermion dynamical symmetry model
  (fermionic counterpart of IBM)
- DJ Rowe et al (1998), A Volya, V Zelevinsky (2003)....: various shell-like models describing the ground-state pairing transition
1) **Phenomenology** (properties of the mean field)
   - Landau theory (1930’s): analysis of the free energy in terms of order parameters
   - Catastrophe theory (1960’s): classification of structurally unstable potentials

2) **Mathematical physics** (mechanism responsible for QPTs)
   - Pechukas-Yukawa approach: Coulomb-gas analogy for level dynamics
   - Branch points: level crossings in complex-extended parameter space
   - Monodromy (integrable systems): singular class of orbits that disallows analytic description

3) **Statistical physics** (relation to thermodynamics)
   - Excited-state (finite-temperature) phase transitions
   - Thermodynamic phase transitions
Part 1/4:

Nuclear “shape phases”
- phenomenology
Geometric model

quadrupole tensor of collective coordinates (2 shape param’s, 3 Euler angles)

...corresponding tensor of momenta

$$ H = \frac{\sqrt{5}}{2K} [\pi \times \pi]^{(0)} + \ldots + \sqrt{5} A [\alpha \times \alpha]^{(0)} - \sqrt{\frac{35}{2}} B [\alpha \times \alpha]^{(2)} + \ldots $$

Potential energy:

$$ V = A(x^2 + y^2) + B(x^3 - 3y^2x) + C(x^2 + y^2)^2 $$

$$ = A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 $$

depends on 2 internal shape variables

$$ x = \text{Re } \alpha_0 \mid_{\text{PAS}} = \beta \cos \gamma $$

$$ y = \sqrt{2} \text{ Re } \alpha_{\pm 2} \mid_{\text{PAS}} = \beta \sin \gamma $$

=> nuclear shapes appear as “phases”
Interacting boson model

\[ H = \sum_{i,j} u_{ij} b_i^+ b_j + \sum_{i,j,k,l} v_{ijkl} b_i^+ b_j^+ b_k b_l \]

\[ b_i^+ = \begin{cases} s_i^+ & \text{s-bosons (l=0)} \\ d_\mu^+ & \mu = -2, \ldots, 2 \text{ d-bosons (l=2)} \end{cases} \]

- “nucleon pairs with \( l = 0, 2 \)”
- “quanta of collective excitations”

Dynamical algebra: \( \textbf{U}(6) \)

...generators: \( G_{ij} = b_i^+ b_j \)

...conserves: \( N = \sum_i b_i^+ b_i \)

Subalgebras: \( \textbf{U}(5), \textbf{O}(6), \textbf{O}(5), \textbf{O}(3), \textbf{SU}(3), [\overline{\textbf{O}(6)}, \overline{\textbf{SU}(3)}] \)

\[ H = k_0 + k_1 C_1[\textbf{U}(5)] + k_2 C_2[\textbf{U}(5)] + k_3 C_2[\textbf{O}(6)] + k_4 C_2[\textbf{O}(5)] + k_5 C_2[\textbf{O}(3)] + k_6 C_2[\textbf{SU}(3)] \]

Dynamical symmetries (extension of standard, invariant symmetries):

\[ \textbf{U}(6) \supset \textbf{U}(5) \supset \textbf{O}(5) \supset \textbf{O}(3) \]

\[ \supset \textbf{O}(6) \supset \textbf{O}(5) \supset \textbf{SU}(3) \]

\[ k_3 = k_6 = 0 \quad \textbf{U}(5) \]

\[ k_1 = k_2 = k_6 = 0 \quad \textbf{O}(6) \]

\[ k_1 = k_2 = k_3 = k_4 = 0 \quad \textbf{SU}(3) \]

[\( \overline{\textbf{O}(6)}, \overline{\textbf{SU}(3)} \)]
The simplest, one-component version of the model, **IBM-1**

Phase diagram for axially symmetric quadrupole deformation

ground-state = minimum of the potential

$E = A\beta^2 + B\beta^3 \cos 3\gamma + \beta^4$

order parameter:

$\beta = 0$ spherical, $\beta > 0$ prolate, $\beta < 0$ oblate.

$\beta_0 \propto \left(\frac{\lambda - \lambda_0}{\lambda_0}\right)^{1/2}$

critical exponent

Landau (1930’s)

Landau (1930’s)
Catastrophe theory

The notion of structural stability:
“Perturb the Hamiltonian and the topology does not change.”

René Thom (1960’s)

Further development:
EC Zeeman, V Arnol’d, I Stewart...


“ground-state energy phase transitions”
classical & quantal models
**Example:**
(from EC Zeeman’s lecture, 1995)

Cusp catastrophe

\[ V = x^4 + ax^2 + bx \]

Germ = \( x^4 \)

Codim (no. of free param’s) = 2

\[ a \propto L, \quad b \propto K \]

many other examples:
physics, biology, medicine,
economy, sociology,
psychology...

**Question:** How does the displacement \( x \) depend

**Answer:** It is a Cusp catastrophe.

Diagram of a beam with load and compression.
Arnol’d classification:

For parameter-dependent families of functions with codimension $\leq 5$ there are just a few generic families to which the others can be reduced.

<table>
<thead>
<tr>
<th>Germ</th>
<th>codim</th>
<th>Universal unfolding</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>1</td>
<td>$x^3 + ux$</td>
<td>fold</td>
</tr>
<tr>
<td>$x^4$</td>
<td>2</td>
<td>$x^4 + ux^2 + vx$</td>
<td>cusp</td>
</tr>
<tr>
<td>$x^5$</td>
<td>3</td>
<td>$x^5 + ux^3 + vx^2 + wx$</td>
<td>swallowtail</td>
</tr>
<tr>
<td>$x^6$</td>
<td>4</td>
<td>$x^6 + ux^4 + vx^3 + wx^2 + tx$</td>
<td>butterfly</td>
</tr>
<tr>
<td>$x^3 + y^3$</td>
<td>3</td>
<td>$x^3 + y^3 + uxy + vx + wy$</td>
<td>hyperbolic umbilic</td>
</tr>
<tr>
<td>$x^3 - xy^2$</td>
<td>3</td>
<td>$x^3 - xy^2 + u(x^2 + y^2) + vx + wy$</td>
<td>elliptic umbilic</td>
</tr>
<tr>
<td>$x^2 + y^4$</td>
<td>4</td>
<td>$x^2y + y^4 + ux^2 + vy^2 + wx + ty$</td>
<td>parabolic umbilic</td>
</tr>
</tbody>
</table>

+ 4 others with codim=5

For codim $>5$ the classification becomes infinite.
Cusp catastrophe: \[ V = x^4 + ax^2 + bx \]

In the shape-phase case we have:

\[ V \approx A\beta^2 - B\beta^3 + \beta^4 \]

By choosing (i) shift of \( V \) and (ii) shift of \( x \) (both depending on \( a,b \)) we can convert \( V \) to the standard cusp form. However,.....
...the relevant sign of $\beta$ is positive, so its negative branches must be disregarded:

mapping of the shape evolution with $|B| = 1, A \in (-\infty, +\infty)$ onto the cusp parameter space
Part 2/4:

Branch points
Ground-state quantum phase transition \((T=0\ \text{QPT})\)

\[
H(\lambda) = H_0 + \lambda V = (1-\lambda)H(0) + \lambda H(1)
\]

\[
\lambda \in [0,1] \quad [H_0, V] \neq 0 
\]

The ground-state energy \(E_0\) may be a nonanalytic function of \(\lambda\) (for \(\text{dim} \rightarrow \infty\)).

\[
\frac{d}{d\lambda} E_0(\lambda) = \langle \Psi_0(\lambda) \vert V \vert \Psi_0(\lambda) \rangle \equiv \langle V \rangle_0
\]

\[
\frac{d^2}{d\lambda^2} E_0(\lambda) = -2 \sum_{i>0} \frac{\left| \langle \Psi_i(\lambda) \vert V \vert \Psi_0(\lambda) \rangle \right|^2}{E_i(\lambda) - E_0(\lambda)} \leq 0
\]

For \(\langle V \rangle_0 \geq 0\) two typical QPT forms:

<table>
<thead>
<tr>
<th>2nd order QPT</th>
<th>1st order QPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\langle V \rangle_0)</td>
<td>(\langle V \rangle_0)</td>
</tr>
<tr>
<td>(0 \quad \lambda)</td>
<td>(0 \quad \lambda)</td>
</tr>
</tbody>
</table>

There are more possibilities and the Ehrenfest classification is not always applicable...
Pairing+Quadrupole Hamiltonian

\[ H(\eta, \chi) = \hbar \omega \left[ \eta n_d - \frac{1-\eta}{N} Q(\chi) \cdot Q(\chi) \right] \]

\( n_d = d^+ \cdot \tilde{d} \)

quadrupole operator \( Q(\chi) = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)} \)

scaling constant \( \hbar \omega = 1 \text{ MeV} \)

control parameters \( \eta, \chi \)

\[ H = H_0 + \eta V \]

\[ H_0 = -\frac{1}{N} Q(\chi) \cdot Q(\chi) \]

\[ V = n_d + \frac{1}{N} Q(\chi) \cdot Q(\chi) \]

ensures that the thermodynamic limit exists: \( N \to \infty \)

symmetry triangle
QPT mechanism?

Hamiltonian \( H(\lambda) = H_0 + \lambda V \)

\( |\psi_i(\lambda)\rangle \) eigenfunctions

\( E_i(\lambda) \) eigenvalues

No-crossing rule: In generic situations, eigenvalues of \( H(\lambda) \) with the same symmetry quantum numbers do not cross: \( E_i(\lambda) \neq E_{i+1}(\lambda) \)

Instead, one can observe avoided crossings of levels. Rapid structural changes of eigenstates take place at these sites:

\[
\left| \langle \psi_i(\lambda) | \psi_i(\lambda + \delta \lambda) \rangle \right|^2 \approx 1 - (\delta \lambda)^2 \sum_{j \neq i} \frac{\left| \langle \psi_i(\lambda) | V | \psi_j(\lambda) \rangle \right|^2}{[E_i(\lambda) - E_j(\lambda)]^2}
\]

Finite-size precursors of the QPTs are connected with avoided crossings of levels involving the ground-state. These crossings become infinitely close as the size of the system asymptotically increases...
QPT mechanism? Multiple avoided crossing of levels

Example

structure I

structure II

g.s.

$\lambda_c$
Branch points

True degeneracies in the complex plane of \( \lambda \). They are determined by the conditions

\[
\det[E - H(\lambda)] = 0
\]

\[
(\partial / \partial E) \det[E - H(\lambda)] = 0
\]

that after elimination yield:

\[
D(\lambda) = \prod_k D_k(\lambda) = \left(-\frac{n(n-1)}{2}\right) \prod_{i < j} [E_j(\lambda) - E_i(\lambda)]^2 = 0
\]

\[
D_k(\lambda) = \prod_{i(\neq k)} [E_i(\lambda) - E_k(\lambda)].
\]

This has \( \frac{n(n-1)}{2} \) complex conjugate pairs of solutions, where \( n \) is dimension of the Hilbert space. For large \( n \) numerical calculations become prohibitively difficult.
The simplest case
(dim=2)
SU(3)-U(5) $\chi = -\frac{\sqrt{7}}{2}$

IBM Hamiltonian

$$H_\chi(\lambda) = (1 - \lambda) \left[ -\frac{Q_x \cdot Q_y}{N} \right] + \lambda n_d$$

with $N = 20$

Branch points for $J=0$

O(6)-U(5) $\chi = 0$

Solution of $\det[E-H(\lambda)]=0 \rightarrow$ single function living on $n$ interconnected **Riemann sheets**

Crossings = complex **square root** type singularities

Branch points can be sorted to $n$ Riemann sheets associated with individual levels for $\lambda$ real

(each sheet contains $n-1$ complex conjugate pairs)

For the ground-state QPT only the ground-state Riemann sheet is relevant.
Excursion to thermodynamic phase transitions (TPTs): Zeros of partition function in the complex-temperature plane

CN Yang, TD Lee (1952)
S Grossmann, W. Rosenhauer, V Lehmann (1967,69) ..... 
P Borrmann et al (2000) ......

Specific heat = an indirect measure of the density of complex zeros close to real $T$
Let us assume that there exists an $\lambda \leftrightarrow T$ analogy between the branch points on the g.s. Riemann sheet and complex zeros of $Z$, namely, let us consider

1) the ground-state partial discriminant $D_0$ being an analog of $\Omega$th power of the partition function (where $\Omega \propto n - 1$):

$$\mathcal{X}^\Omega = \prod_{i>0} (E_i - E_0)$$

2) the 2D “Coulomb-gas energy” being proportional to free energy:

$$\mathcal{U} = - \sum_{i>0} \ln |E_i - E_0|$$

3) the following expression being an analog of specific heat:

$$\mathcal{C} = - \frac{\lambda}{\Omega} \frac{\partial^2 (\lambda \mathcal{U})}{\partial \lambda^2} = \frac{2\lambda}{\Omega} \sum_{i>0} \left[ \frac{\partial^2 E_i}{\partial \lambda^2} - \frac{\partial^2 E_0}{\partial \lambda^2} \right] \left( \frac{\partial E_i}{\partial \lambda} - \frac{\partial E_0}{\partial \lambda} \right)^2 + \frac{dE_i}{E_i - E_0} - \frac{dE_0}{E_0 - E_0}$$

P Cejnar, S Heinze, J Dobeš, PRC 71, 011304(R), (2005).
1) Branch points accumulate close to real-\(\lambda\) axis at QPTs
2) The rate of accumulation is quantitatively the same as the accum. of \(Z=0\) points close to real-\(T\) axis at TPTs !!!

P Cejnar, S Heinze, J Dobeš, PRC 71, 011304(R), (2005).
Part 3/4:
Finite temperatures
**Thermodynamic phase transitions** (temperature-driven)

**Helmholtz free energy**

\[ F_\hat{\rho} (T) = \text{tr}(\hat{\rho} \hat{H}) - T \text{tr}(\hat{\rho} \ln \hat{\rho}) \]

**Thermal population of states**

\[ \hat{\rho}_0 (T) = \frac{1}{Z(T)} \exp(-T^{-1} \hat{H}) \quad Z(T) = \text{tr} \left[ \exp(-T^{-1} \hat{H}) \right] \]

**Equilibrium free energy**

\[ F_0 (T) = -T \ln Z(T) \]

\[ \frac{d}{dT} F_0 (T) = \text{tr}(\hat{\rho}_0 \ln \hat{\rho}_0) \equiv -S_0 \]

\[ \frac{d^2}{dT^2} F_0 (T) = \frac{1}{T^3} \left[ \text{tr}(\hat{\rho}_0 \hat{H}^2) - \text{tr}^2(\hat{\rho}_0 \hat{H}) \right] \geq 0 \]

**Entropy** is a nondecreasing function of \( T \)

In the **thermodynamic limit** \( (N \to \infty |_{N/V}) \), the entropy (thus \( F_0 \)) may be **nonanalytic**

![Graph showing thermodynamic phase transitions](image-url)
Thermodynamic phase transition

\[
\frac{dF_0}{dT} = -S_0, \\
\frac{d^2 F_0}{dT^2} = -\frac{\langle E^2 \rangle_T - \langle E \rangle_T^2}{T^3} \equiv -\frac{C_V}{T}
\]

Quantum phase transition \((T=0)\)

\[
H(\lambda) = H_0 + \lambda V = (1-\lambda)H(0) + \lambda H(1)
\]

\[
\lambda \in [0, 1], \quad [H_0, V] \neq 0
\]

\[
\begin{align*}
\frac{dE_0}{d\lambda} &= \langle \Psi_0 | V | \Psi_0 \rangle \equiv \langle V \rangle_0, \\
\frac{d^2 E_0}{d\lambda^2} &= -2 \sum_{i>0} \frac{|\langle \Psi_i | V | \Psi_0 \rangle|^2}{E_i - E_0}
\end{align*}
\]

What happens at \(T>0\)?
Finite-$T$ extension of a quantum phase transition

**Possibility 1:** isolated QPT at $T=0$

... nongeneric situation

Example:

Isolated structural phase transition at $T=0$

Unavoided crossing of the lowest states
Finite-$T$ extension of a quantum phase transition

**Possibility 2:** QPT survives inclusion of finite $T$

... expected situation

Continuous phase separatrix in the $(\lambda, T)$ plane

[with possible endpoint $(\lambda_t, T_t)$]

\[
\frac{\partial}{\partial \lambda} F_0(\lambda, T) = \frac{\partial^2}{\partial \lambda^2} F_0(\lambda, T) = ?
\]

\[
\frac{\partial}{\partial T} F_0(\lambda, T) = -S_0 \quad \frac{\partial^2}{\partial T^2} F_0(\lambda, T) = \frac{1}{T^3} \langle \Delta^2 E \rangle_T
\]
**Finite-\(T\) extension of a quantum phase transition**

**Thermodynamic phase transition**

\[
\begin{align*}
\frac{dF_0}{dT} &= -S_0, \\
\frac{d^2 F_0}{dT^2} &= -\frac{\langle E^2 \rangle_T - \langle E \rangle_T^2}{T^3} \equiv -\frac{C_V}{T}
\end{align*}
\]

**Quantum phase transition (\(T=0\))**

\[
\begin{align*}
\frac{dE_0}{d\lambda} &= \langle \Psi_0 | V | \Psi_0 \rangle \equiv \langle V \rangle_0, \\
\frac{d^2 E_0}{d\lambda^2} &= -2 \sum_{i>0} \frac{|\langle \Psi_i | V | \Psi_0 \rangle|^2}{E_i - E_0}
\end{align*}
\]

**“Excited-state quantum phase transition” (\(T>0\))**

\[
\begin{align*}
\left. \frac{\partial F_0}{\partial \lambda} \right|_T &= \sum_i \frac{e^{-\frac{E_i}{kT}}}{Z} \langle \Psi_i | V | \Psi_i \rangle \equiv \langle V \rangle_T \\
\left. \frac{\partial^2 F_0}{\partial \lambda^2} \right|_T &= -\frac{1}{T} \sum_i \left[ \frac{e^{-\frac{E_i}{kT}}}{Z} \left( \langle V \rangle_i - \langle V \rangle_T \right)^2 \right] + \sum_i \left[ \frac{e^{-\frac{E_i}{kT}}}{Z} \frac{2}{E_i - E_j} \sum_{j(\neq i)} |\langle \Psi_i | V | \Psi_j \rangle|^2 \right] \\
&= -\frac{1}{T} \left\langle \left( \langle V \rangle_i - \langle V \rangle_T \right)^2 \right\rangle_T + \left\langle \frac{d^2 E_i}{d\lambda^2} \right\rangle_T
\end{align*}
\]
Finite-$T$ extension of a quantum phase transition

Nonzero density of $Z=0$ points as $\text{Im} T \to 0$ (thermodynamic limit)

Nonzero density of branch points as $\text{Im} \lambda \to 0$ (thermodynamic limit)

Surmise:

$$
\rho_{\text{br}}^{(T)}(\lambda) = \sum_i \frac{e^{-T^{-1}E_i(\lambda)}}{Z(\lambda, T)} \rho_{\text{br}}^{(i)}(\lambda)
$$

density on the $i$th Riemann sheet,
or:

$$
\lim_{\text{Im} \lambda \to 0} \rho_{\text{br}}^{(i)}(\lambda), \quad \lim_{\text{Im} \lambda \to 0} \frac{\partial}{\partial \text{Im} \lambda} \rho_{\text{br}}^{(i)}(\lambda)
$$

$T > 0$ relevant density
Monodromy and excited-state QPTs in integrable systems
N=80
all levels with J=0

What about “phase transitions” for excited states (if any) ???
in the sense of nonanalytic evolutions of $E_i(\lambda)$ and $|\psi_i(\lambda)\rangle$

ground-state phase transition (2nd order)
**General analytic approach:** To employ the basis (condensate of $N-k$ bosons) plus ($k$ bosons in single-particle states) with $k=0...N$ and evaluate the dependence of eigenvalues on $\lambda$. (Work in progress: F Iachello, M Caprio...)

**Approach discussed here:** based on some analogies in classical mechanics (monodromy) and on some specific analytic approximations (shifted oscillator approximation). Not systematic and applicable only in integrable systems [i.e., along $O(6)-U(5)$ in the IBM], nevertheless still yielding some new insights...

*N=80 all levels with J=0*

**Ground-state phase transition (2nd order)**
### Integrable systems

**Hamiltonian for** \( f \) **degrees of freedom:**

\[
H(x_i, p_i)
\]

\[x_i \equiv (x_1, x_2, \ldots, x_f)\]

\[
p_i \equiv (p_1, p_2, \ldots, p_f)
\]

**\( f \) integrals of motions “in involution” (compatible):**

\[
C_k(x_i, p_i) \quad k = 1 \ldots f
\]

\[
\{H, C_k\} = 0, \quad \{C_k, C_l\} = 0
\]

**Action-angle variables:**

\[
\begin{align*}
\{x_i\} & \rightarrow \left\{ A_i \right\} \\
P_i & \rightarrow \left\{ I_i \right\}
\end{align*}
\]

\[
\frac{d}{dt} A_i = \omega_i
\]

\[
\frac{d}{dt} I_i = 0
\]

\[
A_i(t) = \omega_i t + A_i(0)
\]

\[
I_i(t) = \text{const}
\]

The motions in phase space stick onto surfaces that are topologically equivalent to **tori**
Monodromy in classical and quantum mechanics

Etymology: Μονοδρομια = “once around”


Simplest example: spherical pendulum

Hamiltonian \( H = \frac{1}{2} \left( p_x^2 + p_y^2 + p_z^2 \right) + z \)

Constraints \( x^2 + y^2 + z^2 = 1 \)
\( xp_x + yp_y + zp_z = 0 \)

Conserved angular momentum: \( L_z = xp_y - yp_x \)

\( \{H, L_z\}_{\text{Poisson}} = 0 \quad \Rightarrow \quad 2 \text{ compatible integrals of motions, } 2 \text{ degrees of freedom} \) (integrable system)
Singular bundle of orbits: trajectories with $E=1, L_z=0$

“pinched torus”

point of unstable equilibrium (trajectory needs infinite time to reach it)

…corresponding lattice of quantum states:
It is impossible to introduce a global system of 2 quantum numbers defining a smooth grid of states:

\( q.\text{num.}\#1: \) z-component of ang.momentum \( m \)
\( q.\text{num.}\#2: \) ??? candidates: “principal.q.num.” \( n \), “ang.momentum” \( l \), combination \( n+m \)

“crystal defect” of the quantum lattice

Another example: **Mexican hat** (champagne bottle) potential

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) - a \rho^2 + b \rho^4 \]


Pinched torus of orbits: \( E=0, L_z=0 \)

Radial q.num. \( n_{\text{rad}} \)

Principal q.num. \( 2n_{\text{rad}} + m \)

Crystal defect of the quantum lattice
The O(6)-U(5) transitional system is integrable: the O(5) Casimir invariant remains an integral of motion all the way and seniority \( \nu \) is a good quantum number.

Classical limit for \( J=0 \):

\[
\frac{H}{N} \rightarrow H_{cl} = \frac{\eta}{2} J^2 + (1-\eta) \beta^2 \pi^2 + \frac{5\eta-4}{2} \beta^2 + (1-\eta) \beta^4
\]

- kinetic energy \( T_{cl} \)
- potential energy \( V_{cl} \)

\[
\beta^2 = x^2 + y^2 \\
\pi^2 = \pi_x^2 + \pi_y^2 = \pi_\beta^2 + \left(\frac{\pi_\gamma}{\beta}\right)^2
\]

\( \beta \in [0, \sqrt{2}] \), \( \pi_\beta \in [0, \sqrt{2}] \), \( \pi_\gamma \in [0,1] \)

\( \nu(\nu+3) \) “seniority”

J=0 projected O(5) “angular momentum”
O(6)-U(5) transition

\[ \frac{1}{N} H = -\frac{1}{N^2} (d^+ s + s^+ \tilde{d}) \cdot (d^+ s + s^+ \tilde{d}) \]

\[ H_{cl} = \beta^2 \pi^2 - \frac{4}{2} \beta^2 + \beta^4 \]

\[ \frac{1}{N} H = \frac{1}{N} n_d \]

\[ H_{cl} = \frac{1}{2} \pi^2 + \frac{1}{2} \beta^2 \]

O(6) \quad \eta \quad U(5)
Available phase-space volume at given energy

\[ \Omega(E) = \int \delta(E - H(p,q)) \, dp \, dq \]

connected to the smooth component of quantum level density

\[ \rho(E) = \text{Tr} \, \delta(E - \hat{H}) = \rho_{\text{smooth}}(E) + \rho_{\text{oscillatory}}(E) \]

\[ \propto \Omega(E) \]

Volume of the “enveloping” torus:

\[ \Omega(E) = \int_{\beta_{\text{min}}(E)}^{\beta_{\text{max}}(E)} 2\pi\beta \, k(\beta; E) \, d\beta \]

\[ \propto 2p_{\beta}^{\text{max}}(\beta; E) \]

P Cejnar et al., subm.to J.Phys.A
Classification of trajectories by the ratio \( R = \frac{T_\gamma}{T_\beta} = \frac{\omega_\beta}{\omega_\gamma} \) of periods associated with oscillations in \( \beta \) and \( \gamma \) directions. For rational \( R = \frac{\mu_\beta}{\mu_\gamma} \) the trajectory is periodic:

\[ M \text{ Macek, P Cejnar, J Jolie, S Heinze, Phys. Rev. C 73, 014307 (2006).} \]
At $E=0$ the motions change their character from O(6)- to U(5)-like type of trajectories.
Lattice of $J=0$ states
$(N=40)$

$J=0$ level dynamics across the O(6)-U(5) transition (all $\nu$'s)

$N=40$

Wave functions in an oscillator approximation:


Method applicable along O(6)-U(5) transition for $N \to \infty$ and states with rel.seniority $v/N=0$:

\[
x = \frac{2n_d}{N} - 1, \quad \langle n_d | \Psi_i \rangle \equiv \psi_i(x)
\]

$x$ may be treated as a continuous variable ($N\to\infty$)

\[
H | \Psi_i \rangle = E_i | \Psi_i \rangle
\]

\[
\langle n_d | H | n_d \rangle \psi_i(x) + \langle n_d | H | n_d+2 \rangle \psi_i(x + \frac{1}{N}) + \langle n_d | H | n_d-2 \rangle \psi_i(x - \frac{1}{N}) = E_i \psi_i(x)
\]

\[
\psi_i(x \pm \frac{1}{N}) = \psi_i(x) \pm \frac{1}{N} \frac{d}{dx} \psi_i(x) + \frac{1}{2N^2} \frac{d^2}{dx^2} \psi_i(x)
\]

$H \xrightarrow{N \to \infty}$ oscillator with $x$-dependent mass:

\[
H_{osc} = -\frac{d}{dx} K(x) \frac{d}{dx} + L(x-x_0)^2 + E_0
\]

O(6) quasi-dynamical symmetry breaks down once the edge of semiclassical wave function reaches the $n_d=0$ or $n_d=N$ limits.
For $v=0$ eigenstates of 

$$\frac{H(\eta)}{N} = \frac{\eta}{N} n_d - \frac{1-\eta}{N^2} Q(0) \cdot Q(0)$$

we obtain:

$$\frac{1}{1-\eta} H_{\text{osc}}(\eta) = \frac{3\hbar^2}{4} \frac{d}{d^2} \left( 1 - x^2 \right) \frac{d}{dx} + \left[ x - \frac{\eta}{4(\eta+1)} \right]^2 - \left[ \frac{5\eta - 4}{4} \right]^2 \propto E_0(\eta)$$

At $E=0$ all $v=0$ states undergo a nonanalytic change.

$V_\eta(x) \leq 0$

$x = \frac{2n_d}{N} - 1 \Rightarrow x \in [-1, +1] \equiv [x_{\text{min}}, x_{\text{max}}]$  

$\Rightarrow$ approximation holds for energies below $E_{\text{up}}(\eta) = V_\eta(x_{\text{min}}) = 0$

P Cejnar et al., subm.to J.Phys.A
**x-dependence of velocity**

( classical limit of $|\psi(x)|^2$ )

<table>
<thead>
<tr>
<th>$E = 0$</th>
<th>$E &lt; 0$</th>
</tr>
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Effect of $m(x) \to \infty$ for $x \to -1$

Similar effect appears in the $\beta$-dependence of velocity$^{-1}$ in the Mexican hat at $E=0$

$x = -1 \iff n_d = 0 \iff \beta = 0$

In the $N \to \infty$ limit the average $<n_d> \to 0$ (and $<\beta> \to 0$) as $E \to 0$.

At $E=0$ all $v=0$ states undergo a nonanalytic change.

P Cejnar et al., subm.to J.Phys.A
\[ S_{i}^{U(5)} = -\sum_{n_d} P_i(n_d) \ln P_i(n_d) \]

U(5) wave-function entropy

\( |\Psi(n_d)|^2 \)

\( u = 0 \)

\( E_{\text{up}} = 0 \)

Any phase transitions for **non-zero** seniorities?

\[
\frac{1}{N} H \xrightarrow{J = 0} H_{cl} = \left[ \frac{\eta}{2} + (1 - \eta) \beta^2 \right] \left[ \pi_\beta^2 + \left( \frac{\pi_\gamma}{\beta} \right)^2 \right] + \frac{5\eta - 4}{2} \beta^2 + (1 - \eta) \beta^4
\]

\[
\frac{1}{N^2} C_2[O(5)] \leftrightarrow \left( \frac{\nu}{N} \right)^2 \equiv \delta^2 \xrightarrow{J = 0} \pi_\gamma^2
\]

\[H_{cl} = \left[ \frac{\eta}{2} + (1 - \eta) \beta^2 \right] \pi_\beta^2 + (1 - \eta) \delta^2 + \frac{\eta \delta^2}{2 \beta^2} + \frac{5\eta - 4}{2} \beta^2 + (1 - \eta) \beta^4 \]

constant & centrifugal terms

\[V_{\text{eff}}(\beta) = \frac{64444444448}{14444444443} \]

\[\beta_0 = 1\]

\[\sqrt{\delta}
\]

For \(\delta \neq 0\) fully analytic evolution of the minimum \(\beta_0\) and min. energy \(V_{\text{eff}}(\beta_0)\)

\[\Rightarrow \textbf{no phase transition} !!!\]

P Cejnar et al., subm.to J.Phys.A
$J=0$ level dynamics for separate seniorities

$N=80$

$E_{up} = 0$

excited states

continuous

(maybe with no Ehrenfest classif.)

ground state

2nd order

no phase transition

$\nu=0$

$\nu=18$

P Cejnar et al., subm. to J.Phys.A
Surprising analogy:

M Macek et al, in preparation

O(6)-U(5) “semi-regular arc”

Does this imply similar QPT behavior in the arc? How to define monodromy in nonintegrable regimes? Phase transitions for other paths in the triangle?

(Alhassid et al, 1991)
Collaborators: Michal Macek, Pavel Stránský (Prague)
Jan Dobeš (Řež)
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Zdeněk Pluhař (Prague)

.................
APPENDICES
Quantum lattice of states: **U(5) limit**

\[
\frac{1}{N} H = \frac{1}{N} n_d
\]

\[
H_{\text{cl}} = \frac{1}{2} \pi^2 + \frac{1}{2} \beta^2
\]

**Analogy with standard isotropic 2D harmonic oscillator:**

**angular-momentum quantum number** \( m \)

**radial quantum number** \( n_{\text{rad}} \)

**principal quantum number** \( N = 2n_{\text{rad}} + m \)

\( n_1 = n_{\text{rad}} + v/3 \)

\( n_2 = n_{\text{rad}} + 2v/3 \)

\( \ldots \text{yes, but only for } n_d = 3k \)

\* differences between the O(2) and \( J=0 \) projected O(5) angular momenta
\( \frac{1}{N} H = -\frac{1}{N^2} (d^+ s + s^+ \tilde{d}) \cdot (d^+ s + s^+ \tilde{d}) \)

\[ H_{cl} = \beta^2 \pi^2 - \frac{4}{2} \beta^2 + \beta^4 \]

\( N = 40 \)
Transitional case

\[ \frac{1}{N} H = \frac{n}{N} n_d - \frac{1-n}{N^2} Q(0) \cdot Q(0) \]

\[ H_{cl} = \frac{n}{2} \pi^2 + (1-n) \beta^2 \pi^2 \]
\[ + \frac{5n-4}{2} \beta^2 + (1-n) \beta^4 \]

\[ \frac{1}{2} (\frac{V}{(V+3)})^{\frac{1}{2}} \]

\[ \frac{N}{2} \]

\[ \frac{N}{2} \]

\[ M_{\text{Macek}}, P \text{ Cejnar}, J \text{ Jolie}, S \text{ Heinze}, \]
\[ \text{Phys. Rev. C 73, 014307 (2006).} \]
$J=0$ level dynamics across the O(6)-U(5) transition (separate $\nu$'s)

Berry-Tabor trace formula
(an analog of the Gutzwiller formula, but for 2D integrable systems)

$$
\rho_{\text{fluct}}(E) = \frac{1}{\pi \hbar} \sum_\mu \sum_{r=1}^\infty \frac{T_\mu}{\sqrt{\hbar (r \mu_2)^3}} \cos \left[ \frac{1}{\hbar} r S_\mu(E) - \frac{\pi}{2} r \nu_\mu - \frac{\pi}{4} \right]
$$

- $\rho_{\text{fluct}}(E)$ — fluctuating part of level density
- $\mu = (\mu_1, \mu_2)$ — pair of integers characterizing periodic orbit with ratio of frequencies $\frac{\omega_1}{\omega_2} = \frac{\mu_1}{\mu_2}$
- $r$ — number of repetitions
- $T_\mu$ — period of the primitive orbit
- $g_E(I_1)$ — function defined by $H(I_1, I_2 = g_E) = E \Rightarrow g_E^* = -\frac{\omega_1}{\omega_2}$
- $S_\mu(E) = 2\pi (I \cdot \mu)$ — action per period
- $\nu_\mu$ — Maslov index of the primitive orbit
average numbers of d-bosons

**IBM classical limit**

General method: **coherent states** (Schrödinger, Glauber, Gilmore, Perelomov,....)


- **use of Glauber coherent states**
  \[
  |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \exp(\alpha_s s^+ + \sum_{\mu} \alpha_\mu d^+) |0\rangle
  \]

- **classical Hamiltonian**
  \[
  H_{cl} = \langle \alpha | H | \alpha \rangle
  \]

- **boson number conservation (only in average)**
  \[
  \langle N \rangle = \langle \alpha | N | \alpha \rangle = |\alpha_s|^2 + \sum_{\mu} |\alpha_\mu|^2
  \]

- **classical limit:**
  \[
  \langle N \rangle \rightarrow \infty
  \]
  \[
  \alpha \rightarrow \frac{\alpha}{\sqrt{\langle N \rangle}} = \frac{1}{\sqrt{2}} \left[ (-)^\mu q_\mu + ip_\mu \right]
  \]
  \[
  \sum_{\mu} \left( p_\mu^2 + q_\mu^2 \right) \leq 2
  \]

- **angular momentum \(J=0\)**
  \[
  \Rightarrow \text{Euler angles irrelevant} \Rightarrow \text{only 4D phase space}
  \]
  \[
  2 \text{ coordinates } (x, y) \text{ or } (\beta, \gamma)
  \]

- **result:**
  \[
  T = \frac{1}{2K} \left[ \pi_\beta^2 + \left( \frac{\pi_\gamma}{\beta} \right)^2 \right] + f(\beta, \gamma, \pi_\beta, \pi_\gamma)
  \]
  \[
  V = A\beta^2 + B\beta^3 \sqrt{1 - \frac{\beta^2}{2}} \cos 3\gamma + C\beta^4
  \]

Similar to GCM but with position-dependent kinetic terms and higher-order potential terms

\[
\beta \in [0, \sqrt{2}]
\]