Exceptional Points and Quantum Phase Transitions

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Stellenbosch 2010
Exceptional Points and Quantum Phase Transitions

Lecture II
Exceptional Points – hidden machinery of quantum phase transitions

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Exceptional Points – hidden machinery of quantum phase transitions

1. Level crossings, avoided level crossings, exceptional points
   • Avoided level crossings and their link to the (ES)QPTs
   • Non-Hermitian degeneracies: typology, Riemann sheet structure, computational difficulties...

2. Distribution of EPs near quantum critical points
   • A method to indirectly measure the relevant EP density close to the real parameter axis (2D Coulomb analogy)
   • Results near (ES)QPTs in some simple models
   • Analogy between the EPs and the Yang-Lee zeros of a partition function ("thermodynamic analogy" for QPTs)
No-crossing “theorem”  

Hamiltonian \( H(\lambda) = H_0 + \lambda H' \) with eigensolutions \( E_i(\lambda) \) and \( |\psi_i(\lambda)\rangle \)

**Statement:** In “generic situations”, eigenvalues of \( H(\lambda) \) within the same symmetry subspace do not cross: \( E_i(\lambda) \neq E_{i+1}(\lambda) \)

\[
\det(E-H) = 0
\]
**No-crossing “theorem”**

Hamiltonian  \( H(\lambda) = H_0 + \lambda H' \) with eigensolutions \( E_i(\lambda) \) and \( |\psi_i(\lambda)\rangle \)

**Statement:** In “generic situations”, eigenvalues of \( H(\lambda) \) within the same symmetry subspace do not cross: \( E_i(\lambda) \neq E_{i+1}(\lambda) \)

This maximum goes either up or down* with increasing \( \lambda \)

\[ \Rightarrow \text{The corresponding pair of solutions disappears on one side of the degeneracy.} \]

But this cannot happen!

* The maximum may just touch the plane, but this is unlikely except if the two states belong to two different non-interacting (e.g., symmetry-classified) subspaces of the Hilbert space.

[von Neumann, Wigner 1929]
No-crossing "theorem" [von Neumann, Wigner 1929]

Hamiltonian $H(\lambda) = H_0 + \lambda H'$ with eigensolutions $E_i(\lambda)$ and $|\psi_i(\lambda)\rangle$

**Statement:** In “generic situations”, eigenvalues of $H(\lambda)$ within the same symmetry subspace do not cross: $E_i(\lambda) \neq E_{i+1}(\lambda)$

$\Rightarrow$ Hence no real crossings within the same symmetry subspace, just “avoided crossings”
Avoided crossings

Rapid structural changes of eigenstates at these sites:

\[
\left| \langle \psi_i(\lambda) | \psi_i(\lambda + \delta \lambda) \rangle \right|^2 \approx 1 - (\delta \lambda)^2 \sum_{j \neq i} \left| \frac{\langle \psi_i(\lambda) | H | \psi_j(\lambda) \rangle}{[E_i(\lambda) - E_j(\lambda)]^2} \right|^2
\]

In a binary avoided crossing, the two levels involved swap the wave functions (closely related to the Landau-Zener mechanism, 1932)

\[
|\psi_{\text{up}}(-\infty)\rangle = |\psi_{\text{low}}(+\infty)\rangle
\]

\[
|\psi_{\text{low}}(-\infty)\rangle = |\psi_{\text{up}}(+\infty)\rangle
\]

They seem to play an important role in QPTs and ESQPTs!
Avoided crossings

Close to each avoided crossing there is a real crossing in complex $\lambda$

non-Hermitian degeneracy
Non-Hermitian degeneracies

**exceptional (branch) point**
complex-square-root geometry

\[ \Delta E \propto -\sqrt{\Delta \lambda} \]

\[ \text{Re} \Delta E = 0 \]
\[ \text{Im} \Delta E = 0 \]

...branch point...

**diabolical point**
conical geometry

...diabolical point...

*Berry, Wilkinson (1984)*

**Generic type** of non-Hermitian degeneracy

**Exceptional type** of non-Hermitian degeneracy for the Hamiltonians linear in \( \lambda \)
Non-Hermitian degeneracies

finite dimension $n$

Eq.1

$$\text{det}[E - H(\Lambda)] = 0$$

polynomial of order $n$ in $E & \Lambda$

Eq.2

$$\frac{\partial}{\partial E} \text{det}[E - H(\Lambda)] = 0$$

polynomial of order $(n-1)$ in $E & \Lambda$

$$\Rightarrow \quad D(\Lambda) = (-)^{\frac{n(n-1)}{2}} \prod_{i < j} [E_j(\Lambda) - E_i(\Lambda)]^2 = \prod_k D_k(\Lambda) = 0$$

Discriminant

$D(\Lambda) = \text{polynomial of order } n(n-1) \text{ with } n(n-1)/2 \text{ complex-conjugate pairs of roots (crossings of each level with all the others)}$

Partial discriminant

$$D_k(\Lambda) = \prod_{i(\neq k)} [E_i(\Lambda) - E_k(\Lambda)]$$

Complex solutions of Eq.1 in the complex plane $\Lambda$ live on a system of $n$ Riemann sheets which are pairwise connected at the points of non-Hermitian degeneracies given as $D(\Lambda)=0$. 
Non-Hermitian degeneracies: Riemann sheets

Riemann sheets of $\det[E - H(\Lambda)] = 0$ can be enumerated according to the ordering of solutions on the real axis. For the dynamics of level $k$ only the degeneracies on $k^{th}$ sheet are relevant. But a numerical assignment of degeneracies to individual sheets is very difficult!

**Example:**

**IBM** along O(6)-U(5)

$$H = \eta \frac{n_d}{N} - \frac{1-\eta}{N^2} Q_0 \cdot Q_0$$

$$Q_0 = d^+ s + s^+ d$$
Non-Hermitian degeneracies: Riemann sheets

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Example: IBM along $O(6)$–$U(5)$

\[
H = \frac{\eta}{N} n_d - \frac{1-\eta}{N^2} Q_0 \cdot Q_0
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Non-Hermitian degeneracies: Riemann sheets
Non-Hermitian degeneracies: Coulomb analogy

Riemann sheets of $\det[E-H(\Lambda)]=0$ can be enumerated according to the ordering of solutions on the real axis. For the dynamics of level $k$ only the degeneracies on $k^{th}$ sheet are relevant. But a numerical assignment of degeneracies to individual sheets is very difficult!

We seek for the roots of the $k^{th}$ partial discriminant:

$$D_k(\Lambda) = \prod_{i(\neq k)} [E_i(\Lambda) - E_k(\Lambda)]$$

$$[D_k(\Lambda)]^2 = c \prod_{j=1}^{n-1} (\Lambda - \Lambda_{kj})(\Lambda - \overline{\Lambda}_{kj})$$

On the real axis we have:

$$[D_k(\lambda)]^2 = c \prod_{j=1}^{n-1} [(\lambda - \lambda_{kj})^2 + \mu_{kj}^2]$$

$$\Lambda_{kj} = \lambda_{kj} + i\mu_{kj}$$

the $j^{th}$ root of $D_k^2$

To convert $\Pi \to \Sigma$ we define

$$U_k(\lambda) = -\frac{K}{2} \ln[D_k(\lambda)]^2$$

normalization factor $K \propto \mathbb{N}^{-d}$, first we take $K=1$. 
Non-Hermitian degeneracies: Coulomb analogy

\[ U_k(\lambda) = \text{const} - K \sum_{j=1}^{n-1} \ln R_{kj}(\lambda) = -K \sum_{i(\neq k)} \ln |E_i(\lambda) - E_k(\lambda)| \]

Potential energy of 2D Coulomb gas \((V \sim -\ln R)\); charges = EPs on the \(k\)th sheet. Measures distribution of EPs in \(C\).

Define quantities:

\[ F_k(\lambda) = -\frac{d}{d\lambda} U_k(\lambda) \quad \text{“force”} \]
\[ C_k(\lambda) = \frac{d}{d\lambda} F_k(\lambda) \quad \text{“force gradient”} \]
\[ Q_k(\lambda) = \lim_{\varepsilon \to 0^+} \int_{\lambda - \varepsilon}^{\lambda + \varepsilon} C_k(\lambda') d\lambda' = \lim_{\varepsilon \to 0^+} \left[ F_k \right]_{\lambda - \varepsilon}^{\lambda + \varepsilon} \quad \text{“force discontinuity”} \]

\(Q_k(\lambda) = 0\) \(\Rightarrow\) zero \(\Rightarrow\) nonzero \(\quad \text{“charge density” at } \lambda\)
Non-Hermitian degeneracies: Coulomb analogy

$U_k(\lambda) = \text{const} - K \sum_{j=1}^{n-1} \ln R_{kj}(\lambda) = -K \sum_{i\neq k} \ln |E_i(\lambda) - E_k(\lambda)|$

Potential energy of **2D Coulomb gas** \((V \sim -\ln R)\); **charges = EPs** on the \(k^{\text{th}}\) sheet. Measures distribution of EPs in \(C\).

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$$Q_k(\lambda) = \lim_{\varepsilon \to 0^+} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} C_k(\lambda') d\lambda' = \lim_{\varepsilon \to 0^+} \left[ F_k \right]_{\lambda-\varepsilon}^{\lambda+\varepsilon} \quad \text{"force discontinuity"}$$

**Example: linear arrangement of EPs**

\[ Q_k(\lambda_c) \neq 0 \quad \alpha = 0 \quad \text{EP density near } C \]

\[ Q_k(\lambda_c) = 0 \quad \alpha > 0 \]

\[ C(\lambda_c) = \infty \quad \alpha \in (0,1] \]

\[ \frac{d^2}{dx^2} C(\lambda_c) = \infty \quad \alpha \in (1,3] \]

\[ \frac{d^4}{dx^4} C(\lambda_c) = \infty \quad \alpha \in (3,7] \text{...} \]
Non-Hermitian degeneracies: Coulomb analogy

\[ U_k(\lambda) = \text{const} - K \sum_{j=1}^{n-1} \ln R_{kj}(\lambda) = -K \sum_{i \neq k} \ln \left| E_i(\lambda) - E_k(\lambda) \right| \]

Potential energy of 2D Coulomb gas \((V \sim -\ln R)\); charges = EPs on the \(k\)th sheet. Measures distribution of EPs in \(C\).

Define quantities:

- \(F_k(\lambda) = -\frac{d}{d\lambda} U_k(\lambda)\) \text{ “force”}
- \(C_k(\lambda) = \frac{d}{d\lambda} F_k(\lambda)\) \text{ “force gradient”}
- \(Q_k(\lambda) = \lim_{\varepsilon \to 0^+} \int_{\lambda - \varepsilon}^{\lambda + \varepsilon} C_k(\lambda')d\lambda' = \lim_{\varepsilon \to 0^+} \left[ F_k \right]_{\lambda - \varepsilon}^{\lambda + \varepsilon}\) \text{ “force discontinuity”}

\[ Q_k(\lambda) = 0 \implies \text{zero} \quad \text{“charge density” at } \lambda \]
\[ Q_k(\lambda) \neq 0 \implies \text{nonzero} \]

\[ K \propto N^{-d} \] to suppress discontinuous derivatives (if any) of \(i \neq k\) levels

\[ \sum_{i \neq k} \left( \pm \frac{\dot{E}_i - \dot{E}_k}{E_i - E_k} \right) \]

\[ = K \lim_{\varepsilon \to 0^+} \left[ \frac{d}{d\lambda} \sum_{i \neq k} \ln \left| E_i(\lambda) - E_k(\lambda) \right| \right]_{\lambda - \varepsilon}^{\lambda + \varepsilon} \]

\( \implies \text{continuous} \]
\( \implies \text{discontinuous} \) \(k\)th level derivative at \(\lambda\)
2 x 2 example: \[ H = H_0 + \lambda H' \quad \lambda \rightarrow \Lambda \equiv \lambda + i\mu \]

\[
H = \begin{pmatrix} e_1 & v \\ v & e_2 \end{pmatrix} + \Lambda \begin{pmatrix} e'_1 & v' \\ v' & e'_2 \end{pmatrix} = \begin{pmatrix} e_1 + \Lambda e'_1 & v + \Lambda v' \\ v + \Lambda v' & e_2 + \Lambda e'_2 \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}
\]

\[ E_\pm = \frac{1}{2} \left[ E_1 + E_2 \pm \Delta E \right] \quad \Delta E = \sqrt{(E_1 - E_2)^2 + 4V^2} = \sqrt{a + b\Lambda + c\Lambda^2} \]

Degeneracy \( \Delta E = 0 \)
- Real case: \( \Lambda_0 = \lambda_0 \pm 0 \)
- Complex case: \( \Lambda_0 = \lambda_0 \pm i\mu_0 \)

Local behavior close to
- Complex degeneracy \( \Lambda_0 = \lambda_0 \pm i\mu_0 \)
- Real degeneracy \( \Lambda_0 = \lambda_0 \pm 0 \) (if any)

Exceptional point:
- \( \Delta E \propto \sqrt{\Lambda - \Lambda_0} \)
- \( \Delta E \propto \pm (\Lambda - \Lambda_0) \)

Diabolical point:
- \( \Delta E \propto \sqrt{(\lambda - \lambda_0)^2 + \mu_0^2} \) (if any)

Passing a branch point along the real axis:
- Avoided crossing of levels
- Actual crossing

Distance of the branch point from a given place on the real axis
- Exc. point \( \mu_0 = 0.05 \)
- Exc. point \( \mu_0 = 0.01 \)
- Diab. point \( \mu_0 = 0 \)
**2 x 2 example:** \[ H = H_0 + \lambda H' \quad \lambda \rightarrow \Lambda \equiv \lambda + i\mu \]

\[ H = \begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix} + \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \Lambda & \nu \\ \nu & -\Lambda \end{pmatrix} \]

\[ C_0 = K \frac{d^2}{d\lambda^2} \ln R = K \frac{d^2}{d\lambda^2} \ln|E_+ - E_-| = K \frac{\nu^2 - \lambda^2}{(\nu^2 + \lambda^2)^2} \]

\[ \nu \rightarrow 0 \Rightarrow \text{“1st order QPT”} \]

\[ \Delta_c \]

\[ v = 0.01 \quad v = 0.02 \quad v = 0.05 \]

**fwhm \propto \nu**

**Peak area:** \[ \text{fwhm} \times \text{height} \propto \nu^{-1} \propto \Delta_c^{-1} \]

**“Force” discontinuity:** \[ Q_0(\lambda_c) \left. \right|_{\nu \rightarrow 0} \rightarrow \infty \]

**K=1:** (no size parameter)
**Cusp potential:**  
1\textsuperscript{st} order QPT

\[ H = H_0 + \lambda H' \quad \lambda \rightarrow \Lambda \equiv \lambda + i\mu \]

\[ H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

Ground state:

\[ C_0 = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_{0i} = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln|E_i - E_0| \]

**Peak area:**

\[ \text{fwhm} \times \text{height} \propto \exp(aN) \]

"Force" discontinuity:

\[ Q_0(\lambda_c) \xrightarrow{N \rightarrow \infty} \infty \]
Cusp potential: 1\textsuperscript{st} order QPT

\[ H = H_0 + \lambda H' \quad \lambda \to \Lambda \equiv \lambda + i\mu \]

\[ H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

Ground state

\[ C_0 = K \frac{d^2}{d\Lambda^2} \sum_{i>0} \ln R_{0i} = K \frac{d^2}{d\Lambda^2} \sum_{i>0} \ln|E_i - E_0| \]

Peak area:

\[ \text{fwhm} \times \text{height} \propto \exp(a\Xi)/\Xi \]

"Force" discontinuity:

\[ Q_0(\lambda_c) \xrightarrow{\Xi \to \infty} \infty \]
Cusp potential: 1st order QPT

\[ H = H_0 + \lambda H' \quad \lambda \rightarrow \Lambda \equiv \lambda + i\mu \]
\[ H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

“Parity doublets” in the critical (degenerate double well) potential

⇒ 1st order QPT is locally a 2-level process!

\[ C_0 = K \sum_{i>0} \ln R_{0i} = K \sum_{i>0} \ln |E_i - E_0| \]

Peak area: \( \text{fwhm} \times \text{height} \propto \exp(a\infty) / \infty \)

“Force” discontinuity: \( Q_0(\lambda_c) \xrightarrow{\infty} \infty \)

50 levels included
Cusp potential: 2nd order QPT

\[ H = H_0 + \lambda H' \quad \lambda \to \Lambda \equiv \lambda + i\mu \]

\[ H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

\[ -\lambda \to 0 \]

Ground state \( C_0 = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_{0i} = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln|E_i - E_0| \)

\[ \mu \to \Lambda_{0i} \]

50 levels included

\[ K=1: \]

\[ \log \mathcal{N} \equiv \log \left( \sqrt{M / \hbar} \right) \]

"Force" discontinuity: \( Q_0(\lambda_c) \xrightarrow{n \to \infty} \infty \)

Peak area: \( \text{fwhm} \times \text{height} \propto \mathcal{N}^{2/3} \)
Cusp potential: 2nd order QPT

\[ H = H_0 + \lambda H' \quad \lambda \to \Lambda \equiv \lambda + i\mu \]

\[ H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

Ground state

\[ C_0 = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_{0i} = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln|E_i - E_0| \]

Peak area: fwhm × height \(\propto N^{-1/3}\)

"Force" discontinuity: \(Q_0(\lambda_c) \xrightarrow{N \to \infty} 0\)

\[ K = \mathcal{N}^{-1} \]

\[ \log \mathcal{N} \equiv \log \left( \frac{\sqrt{M}}{\hbar} \right) \]
Cusp potential: 2\textsuperscript{nd} order QPT

\[ H = H_0 + \lambda H' \quad \lambda \to \Lambda \equiv \lambda + i\mu \]

\[ H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

Ground state

\[ C_0 = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_{0i} = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln|E_i - E_0| \]

\[ \Rightarrow \quad \text{2\textsuperscript{nd} order QPT is locally a many-level process!} \]

Increased density at the bottom of the critical (quartic) potential

\[ \log \mathcal{N} = \log \left( \sqrt{M / \hbar} \right) \]

Peak area: \( \text{fwhm} \times \text{height} \propto \mathcal{N}^{-1/3} \)

"Force" discontinuity: \( Q_0(\lambda_c) \xrightarrow{\mathcal{N} \to \infty} 0 \)
**sd-IBM** along O(6)-U(5)

*2nd order QPT*

\[
\frac{H}{N} = \eta \frac{n_d}{N} - \frac{1-\eta}{N^2} Q_0 \cdot Q_0 \quad Q_0 = d^s + s^d
\]

Ground state

\[
C_0 = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_{0i} = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln |E_i - E_0|
\]

\[\nu=0, \ l=0\]
\[\Rightarrow d_{\text{eff}} = 1\]
\[\Rightarrow K = N^{-1}\]

\[N \quad \text{fwhm} \times \text{height} \propto N^{-1/3}\]

\[Q_0(\lambda_c) \xrightarrow{N \to \infty} 0\]
sd-IBM along O(6)-U(5)

continuous ESQPTs

\[
\frac{H}{N} = \frac{\eta}{N} n_d - \frac{1-\eta}{N^2} Q_0 \cdot Q_0 \quad \quad Q_0 = d^+ s + s^+ \tilde{d}
\]

excited states

\[
C_k = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_{ki} = K \frac{d^2}{d\lambda^2} \sum_{i>0} \ln |E_i - E_k|
\]

excitation ratio

\[
x = \frac{k}{n} \in [0,1]
\]
Critical-point scaling in IBM-like models
Arias, García-Ramos, Dukelsky, Dusuel, Vidal, Pérez-Bernal et al. (2005-2008)

**sd-IBM along O(6)-U(5)**

**2nd order QPT**

- height
- width
- height x width

\[ \propto \frac{N^{1+1/3}}{N} = N^{1/3} \]
\[ \propto \sqrt{N^{4/3}} = N^{2/3} \]
\[ \propto N^{-1/3} \]

slower decrease than for the g.s. ⇒ stronger than 2nd order ESQPT

\[ Q_0(\lambda_c) \xrightarrow{N \to \infty} 0 \]
\[ Q_k(\lambda_c) \xrightarrow{N \to \infty} 0 \]
Non-Hermitian degeneracies: (ES)QPT criteria

\[ K \propto \lambda^{-d_{\text{eff}}} \]

\[ U_k(\lambda) = \text{const} - K \sum_{j=1}^{n-1} \ln R_{kj}(\lambda) = -K \sum_{i(\neq k)} \ln |E_i(\lambda) - E_k(\lambda)| \]

\[ F_k(\lambda) = -\frac{d}{d\lambda} U_k(\lambda) \quad C_k(\lambda) = \frac{d}{d\lambda} F_k(\lambda) \quad Q_k(\lambda) = \lim_{\varepsilon \to 0^+} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} C_k(\lambda')d\lambda' = \lim_{\varepsilon \to 0^+} \left[ F_k \right]_{\lambda-\varepsilon}^{\lambda+\varepsilon} \]

- **EP distribution in** \( \lambda \) **on the** \( k^{th} \) **Riemann sheet** can be indirectly probed by the **level dynamics on** \( \lambda \)

- The **EP distribution near** \( \lambda \) **determines the presence & type of a quantum phase transition** of the **\( k^{th} \) level:**
  - EPs come infinitesimally close to \( \lambda \) \( \Rightarrow \) (ES)QPT present at the given place
  - EPs form a “charge layer” across \( \lambda \) \( \Rightarrow \) first order (ES)QPT
  - EPs approach \( \lambda \) with vanishing density \( \Rightarrow \) continuous (ES)QPT
Non-Hermitian degeneracies: thermodynamic analogy

Chen-Ning Franklin Yang (*1922), Tsung-Dao Lee (*1926)

1957 ... parity non-conservation in weak interactions (Nobel prize)

1952 ... thermodynamics of classical phase transitions by means of complex zeros of the partition function

Elaborated by:
► Fisher (1965)
► Grossmann, Rosenhauer, Lehmann (1967-9)
► Biskup et al (2000)
► Borrmann et al (2000)...
► Chomaz, Gulminelli (2003)
► ........

 Canonical partition function  
\[ Z(\beta) = \sum_i e^{-\beta E_i} \text{, inverse temperature } \beta = \frac{1}{kT} \]

 Thermal occupation probabilities  
\[ p_i(\beta) = \frac{1}{Z(\beta)} e^{-\beta E_i} \]

\[ Z(\beta) \text{ cannot be zero for any physical temperature, but it can be (is) zero at some places in the } \beta \text{ plane: } B \equiv \beta + i \gamma. \text{ If these complex zeros get close to } \beta, \text{ a classical phase transition is encountered at the corresponding place.} \]
Non-Hermitian degeneracies: thermodynamic analogy

Quantum mechanical interpretation of complex zeros:

$$Z(B_0) = 0 \quad \ldots \quad B_0 = \beta_0 + i \frac{\tau_0}{\hbar}$$

$$\langle e^{-iH\tau_0/\hbar} \rangle_{T_0 = \frac{1}{kB_0}} = 0$$

$$\langle \Psi_{\text{can}} | e^{-iH\tau_0/\hbar} | \Psi_{\text{can}} \rangle = 0 \quad \ldots \quad | \Psi_{\text{can}} \rangle \equiv \frac{1}{\sqrt{Z(B_0)}} \sum_i e^{-\beta E_i/2} | E_i \rangle$$

survival amplitude

canonical pure state

Criteria for first order and continuous phase transitions:

Density of zeros $\rho \propto (\text{Im } B)^\alpha$

$$\tau_1 \to 0 \quad \alpha = 0 \quad \nu = 0 \quad 1^{\text{st}} \text{ order}$$

$$0 < \alpha < 1 \quad 2^{\text{nd}} \text{ order}$$

$$1 < \alpha \quad \text{higher order}$$
Non-Hermitian degeneracies: thermodynamic analogy

Complex zeros of partition function
thermodynamic phase transitions

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
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<tbody>
<tr>
<td>$Z(T)$</td>
<td>Partition function</td>
</tr>
<tr>
<td>$F(T) = -T \ln Z(T)$</td>
<td>Free energy</td>
</tr>
<tr>
<td>$C(T) = -T \frac{\partial^2}{\partial T^2} F(T)$</td>
<td>Specific heat</td>
</tr>
<tr>
<td>$Q(T) = \lim_{\varepsilon \to 0^+} \int_{T-\varepsilon}^{T+\varepsilon} C(T')dT'$</td>
<td>Latent heat</td>
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Non-Hermitian degeneracies
quantum phase transitions

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<td>$D_k(\lambda)^K = \left( \prod_{i \neq k} \left[ E_i(\lambda) - E_k(\lambda) \right] \right)^K$</td>
<td></td>
</tr>
<tr>
<td>$U_k(\lambda) = -K \ln D_k(\lambda)$</td>
<td></td>
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<tr>
<td>$C_k(\lambda) = - \frac{d^2}{d\lambda^2} U_k(\lambda)$</td>
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<td>$Q_k(\lambda) = \lim_{\varepsilon \to 0^+} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} C_k(\lambda')d\lambda'$</td>
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References:
Cejnar, Heinze, Dobeš, PRC 71, 011304(R) (2005)
Cejnar, Heinze, Macek, PRL 99, 100601 (2007)
I had a feeling once about Mathematics, that I saw it all—Depth beyond depth was revealed to me—the Byss and the Abyss. I saw, as one might see the transit of Venus—or even the Lord Mayor's Show, a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how it happened and why the tergiversation was inevitable: and how the one step involved all the others. It was like politics. But it was after dinner and I let it go!

Winston Chirchill, *My Early life: 1874 - 1904*