Exceptional Points and Quantum Phase Transitions

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Exceptional Points and Quantum Phase Transitions

**Lecture I**
Quantum Phase Transitions – singularities in quantal spectra affecting the ground state & excited states

**Lecture II**
Exceptional Points – hidden machinery of quantum phase transitions
Lecture I
Quantum Phase Transitions – singularities in quantal spectra affecting the ground state & excited states

1. Quantum phase transitions (QPTs)
   • QPTs in lattice-like systems (rough overview)
   • QPTs in collective many-body systems (finite algebraic models)

2. Excited-state quantum phase transitions (ESQPTs)
   • ESQPTs & thermal phase transitions
   • ESQPTs & classical singularities
   • ESQPTs in 1D & 2D examples
“Now, Nina, do you think you could throw something into the sea?”
“I think I could,” replied the child, “but I am sure that Pablo would throw it a great deal further than I can.”
“Never mind, you shall try first.”
Putting a fragment of ice into Nina’s hand, he addressed himself to Pablo: “Look out, Pablo; you shall see what a nice little fairy Nina is! Throw, Nina, throw, as hard as you can.”
Nina balanced the piece of ice two or three times in her hand, and threw it forward with all her strength.
A sudden thrill seemed to vibrate across the motionless waters to the distant horizon, and the Gallian Sea had become a solid sheet of ice!

Jules Verne: Off on a Comet
1877 Hector Sarvedac
Now, Nina, do you think you could throw something into the sea?

"I think I could," replied the child, "but I am sure that Pablo would throw it a great deal further than I can."

"Never mind, you shall try first."

Putting a fragment of ice into Nina’s hand, he addressed himself to Pablo: "Look out, Pablo; you shall see what a nice little fairy Nina is! Throw, Nina, throw, as hard as you can."

Nina balanced the piece of ice two or three times in her hand, and threw it forward with all her strength.

A sudden thrill seemed to vibrate across the motionless waters to the distant horizon, and the Gallian Sea had become a solid sheet of ice! 
first order (1\textsuperscript{st} kind)

- solid-liquid
- liquid-gas $p < p\text{_{crit}}$
- superconductor 1\textsuperscript{st} kind
- Bose-Einstein
- etc.

(2\textsuperscript{nd} kind) continuous

- liquid-gas $p = p\text{_{crit}}$
- ferromagnet
- superfluid
- etc.

disordered

ordered

thermal phase transitions

$\frac{\partial F_0}{\partial T}$ discontinuous

$\frac{\partial F_0}{\partial T}$ continuous

$\Phi$ order parameter

$T$

$T_c$

$\Phi$
The brave new world of the 20th century...

New types of phases & phase transitions

Quantum fluctuations only!

⇒ Quantum Phase Transitions (QPTs)

Ground-state energy

\[ \frac{\partial E_0}{\partial p} \]

- discontinuous at \( p_c \) ⇒ first order
- continuous at \( p_c \) ⇒ continuous
  
  \( (n\text{th order if } \frac{\partial^n E_0}{\partial p^n} \text{ discont.}) \)
Quantum Phase Transitions („lattice systems“)

S Sachdev, Quantum Phase Transitions (Cambridge Univ. Press, 1999)

- infinite lattice-like systems with semi-local interactions
- continuous phase transition of the type order – disorder
- always accompanied by a universal “quantum-critical” domain at \( T > 0 \)
- sometimes followed by a line of thermal phase transitions
- relevant in numerous “new materials”

![Diagram](image-url)
Quantum Phase Transitions („lattice systems“)

- infinite lattice-like systems with local interactions
- continuous phase transition of the type order – disorder
- always accompanied by a universal “quantum-critical” domain at $T>0$
- sometimes followed by a line of thermal phase transitions
- relevant in numerous “new materials”

Example: Ising model in transverse magnetic field

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h \sum_i S_i^x$$

spin-spin interaction between neighboring sites

$$h = 0 \quad |\Psi_0\rangle = \prod_i |\uparrow\rangle_i \text{ or } |\Psi_0\rangle = \prod_i |\downarrow\rangle_i$$

$$h >> h_c \quad |\Psi_0\rangle \approx \prod_i |\rightarrow\rangle_i$$

external field in $x$-direction

$$S_i^x = \frac{1}{2} (S_i^+ + S_i^-)$$

spin flips

LiHoF$_4$
Quantum Phase Transitions („many-body systems“)

DJ Thouless, Nucl. Phys. 22, 78 (1961)
HJ Lipkin, N Meshkov, AJ Glick, Nucl.Phys. 62, 188 (1965)
R Gilmore, Catastrophe Theory for Scientists and Engineers (Wiley, N.Y., 1981)

Mostly associated with the collective dynamics of many-body systems
⇒ finite number of degrees of freedom ⇒ fundamental consequences (see below)
⇒ algebraic language based on dynamical symmetries & generalizations
  phases → “quasi dynamical symmetries” [Rowe et al. 1988…]
  critical points → approx. “critical point solutions” [Iachello 2000…]
  → “partial dynamical symmetries” [Leviatan 1994…]
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Atomic nuclei
  = finite objects
⇒ may show
  only precursors
  of real QPTs

Example:
spherical-deformed transition

Region I

Region II
Quantum Phase Transitions („many-body systems“)

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Lipkin model: SU(2)

1) $N=2j \leq \Omega$ fermions on 2 levels with capacity $\Omega$

\[
\sum a^+_i a^-_i = \frac{1}{2} \sum a^+_i a^+_i + \frac{1}{2} \sum a^-_i a^-_i
\]

\[
J^+_i = \sum a^+_i a^-_i\]

\[
J^-_i = \sum a^-_i a^+_i\]

\[
J^z = -\frac{1}{2} \sum a^+_i a^-_i + \frac{1}{2} \sum a^+_i a^+_i
\]

2) $N=2j$ spin1/2 particles:

\[
J^+_i = \sum S^+_i\]

\[
J^-_i = \sum S^-_i\]

\[
J^z = \sum S^z_i
\]

3) $N=2j$ interacting scalar/pseudoscalar bosons:

\[
J^+_i = t^+ s\]

\[
J^-_i = s^+ t\]

\[
J^z = \frac{1}{2} \left(t^+ t - s^+ s\right)
\]

\[
H = J^z + \lambda \frac{1}{2N} \left(J^+_i + J^-_i\right)
\]

... "collapse of RPA"
... pseudospin system with a QPT
... nonspin systems with both 1st, 2nd order QPTs

B. Bartlett, nucl-th/0305052
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Lipkin model: SU(2)

1) $N=2j \leq \Omega$ fermions on 2 levels with capacity $\Omega$

+ $J_z = -\frac{1}{2} \sum_{i=1}^{\Omega} a_i^- a_i^- + \frac{1}{2} \sum_{i=1}^{\Omega} a_i^+ a_i^+$

- $J_+ = \sum_{i=1}^{\Omega} a_i^+ a_i^-$  $J_- = \sum_{i=1}^{\Omega} a_i^- a_i^+$

2) $N=2j$ spin1/2 particles:

$J_+ = \sum_i S_i^+  \quad J_- = \sum_i S_i^-  \quad J_z = \sum_i S_i^z$

3) $N=2j$ interacting scalar/pseudoscalar bosons:

$J_+ = t^+ s  \quad J_- = s^+ t  \quad J_z = \frac{1}{2} (t^+ t - s^+ s)$

$H = J_z + \lambda \frac{1}{2N} \left( J_+^2 + J_-^2 \right)$

Bloch sphere

phase II  phase I

T>0  T=0

critical for $j \rightarrow \infty$
Quantum Phase Transitions ("many-body systems")

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Interacting Boson Model: U(6)

\[ H = \sum_{i,j} u_{ij} b_i^+ b_j + \sum_{i,j,k,l} v_{ijkl} b_i^+ b_j^+ b_k b_l \]

\[ = \sum_i w_i C[G_i] \]

Dynamical Symmetries
(special classes of Hamiltonians)
=> algebraic solutions, integrability...
=> QPTs based on competing dynamical symmetries

... ”collapse of RPA”
... pseudospin system with a QPT
... nonspin systems with both 1st, 2nd order QPTs

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**Interacting Boson Model: U(6)**

\[ H = \sum_{i,j} u_{ij} b_i^+ b_j + \sum_{i,j,k,l} v_{ijkl} b_i^+ b_j^+ b_k b_l \]
\[ = \sum_i w_i C[G_i] \]

\[ |\Psi_0\rangle \propto (s^+)^N |0\rangle \]
\[ |\Psi_0\rangle \propto (s^+ + \beta_0 d_0^+)^N |0\rangle \]

**Critical for** \[ N \to \infty \]
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... "collapse of RPA"
... pseudospin system with a QPT
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IBM-like models: \( U(n) \)

\[
H = \sum_{i,j} u_{ij} b_i^+ b_j + \sum_{i,j,k,l} v_{ijkl} b_i^+ b_j^+ b_k b_l = \sum_i w_i C[G_i]
\]

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<th>( n )</th>
<th>( f )</th>
<th>model</th>
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</thead>
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<td>(t)</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>(( \tau ))</td>
<td>3</td>
<td>2</td>
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<tr>
<td>1</td>
<td>(p)</td>
<td>4</td>
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<td>(( \pi ))</td>
<td>5</td>
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<tr>
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<td>(( d ))</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>5/2</td>
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<td>7</td>
<td>6</td>
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<td>8</td>
<td>7</td>
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<tr>
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<td>(( \varphi ))</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>(( g ))</td>
<td>10</td>
<td>9</td>
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\( N \to \infty \)
Quantum Phase Transitions ("many-body systems")

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Quantum optical models: \( HW(1) \otimes G \)

\[
H = \omega b^+ b + \omega_0 J_0 + \lambda \frac{1}{\sqrt{4j}} \left[ b^+ J_- + b J_+ + a \left( b^+ J_+ + b J_- \right) \right]
\]

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<td>SU(2)</td>
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<td>( a=0 ) Jaynes-Cummings model</td>
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<td>2</td>
<td>( a=1 ) Dicke model: superradiance</td>
</tr>
<tr>
<td>SU(1,1)</td>
<td>1</td>
<td>( a=0 ) creation/disociac. 2-atom molecules</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( a=1 )</td>
</tr>
</tbody>
</table>

\[
\frac{\langle J_0 \rangle}{j} = \cos 2\vartheta, \quad \frac{\langle b^+ b \rangle}{2j} = \sin^2 \vartheta
\]

\( \vartheta \) critical for \( j, k \rightarrow \infty \)
**Quantum Phase Transitions** („many-body systems“)

**Some general features of finite algebraic models**

**dynamical algebra**

\[ [g_i, g_j] = c_{ijk} g_k \]

**Hamiltonian** (+ observables)

\[ H = H(\{g_i\}) \]

---

**Size parameter** \( \kappa \) consistent with the following requirements:

1) vanishing thermal fluctuation of scaled energy in the thermodynamic limit:
   (canonical \( \rightarrow \) microcanonical ensemble)

\[
\frac{H}{\kappa} = H\left(\left\{ \frac{g_i}{\kappa} \right\}\right) \quad \frac{\langle (E)^2 \rangle}{\langle \frac{E}{\kappa} \rangle} \xrightarrow{\kappa \rightarrow \infty} 0
\]

2) consistent scaling of Hamiltonian

\[
\frac{\langle X^2 \rangle}{\langle \frac{X}{\kappa} \rangle} \equiv \langle X^2 \rangle_T - \langle X \rangle_T^2
\]

\[
\langle X \cdot Y \rangle \equiv \langle X \cdot Y \rangle_T - \langle X \rangle_T \cdot \langle Y \rangle_T
\]

**Consequence:**

scaled generators with vanishing commutators for \( \kappa \rightarrow \infty \)

\[
\left[ \frac{g_i}{\kappa}, \frac{g_j}{\kappa} \right] \xrightarrow{\kappa \rightarrow \infty} 0 \quad \Rightarrow \quad \text{thermodynamic limit} = \text{classical limit}
Excited-State Quantum Phase Transitions (ESQPTs)

\[ \hat{H} = \hat{H}_0 + \lambda \hat{H}' \]

Free energy

\[
F = \operatorname{Tr}(\hat{\rho} \hat{H}) + T \operatorname{Tr}(\hat{\rho} \ln \hat{\rho}) = - T \ln Z - \langle E \rangle
\]

Hamiltonian, canonical density matrix, partition function

\[
\frac{\partial F}{\partial T} = -S, \quad \frac{\partial^2 F}{\partial T^2} = -\frac{\langle E^2 \rangle}{T^3}
\]

entropy
Excited-State Quantum Phase Transitions (ESQPTs)

\[ \hat{H} = \hat{H}_0 + \lambda \hat{H}' \]

**Free energy**

\[ F = \text{Tr}(\hat{\varrho} \hat{H}) + T \text{Tr}(\hat{\varrho} \ln \hat{\varrho}) = -T \ln Z \]

\( \langle E \rangle \) thermal average of energy

entropy

\[ \frac{\partial F}{\partial T} = -S, \quad \frac{\partial^2 F}{\partial T^2} = -\frac{\langle E^2 \rangle}{T^3} \]

Hamiltonian canonical density matrix partition function

control parameter \( \lambda \)

phase separatrix

QPT

\[ E \equiv \varepsilon \]

level density

entropy

thermodynamic limit

\[ S(\varepsilon) \propto \ln[\rho(\varepsilon) d\varepsilon] \]
Excited-State Quantum Phase Transitions (ESQPTs)

\[ \hat{H} = \hat{H}_0 + \lambda \hat{H}' \]

- Hamiltonian
- canonical density matrix

Free energy
\[ F = \frac{\text{Tr}(\hat{\rho}\hat{H})}{\text{Tr}(\hat{\rho})} + T \frac{\text{Tr}(\hat{\rho} \ln \hat{\rho})}{\text{Tr}(\hat{\rho})} = - T \ln Z \]

- thermal average of energy
- entropy

\[ \frac{\partial F}{\partial T} = - S, \quad \frac{\partial^2 F}{\partial T^2} = - \frac{\langle \langle E^2 \rangle \rangle}{T^3} \]

- thermodynamic limit = classical limit
- level density
- phase space volume

\[ S(\varepsilon) \propto \ln[\rho(\varepsilon) d\varepsilon] \propto \ln[\frac{d}{d\varepsilon} \Omega(\varepsilon) d\varepsilon] \]

- phase separatrix
- control parameter \( \lambda \)
Excited-State Quantum Phase Transitions (ESQPTs)

\[ \hat{H} = \hat{H}_0 + \lambda \hat{H}' \]

Free energy
\[ F = \frac{\text{Tr}(\hat{\rho} \hat{H})}{\text{Tr}(\hat{\rho} \ln \hat{\rho})} = -T \ln Z \]

Hamiltonian
phase separatrix

control parameter \( \lambda \)

\[ \frac{\partial F}{\partial T} = -S, \quad \frac{\partial^2 F}{\partial T^2} = -\frac{\langle \langle E^2 \rangle \rangle}{T^3} \]

slope & curvature of individual levels
\[ \dot{E}_i \equiv \frac{d}{d\lambda} E_i, \quad \ddot{E}_i \equiv \frac{d^2}{d\lambda^2} E_i \]

level density
\[ S(\varepsilon) \propto \ln[\rho(\varepsilon)d\varepsilon] \propto \ln[\frac{d}{d\varepsilon} \Omega(\varepsilon)d\varepsilon] \]

thermodynamic limit
\[ \Omega(\varepsilon) \equiv \frac{1}{\sqrt{2\pi}} \int d\varepsilon' e^{-\varepsilon'^2/2} \]

phase space volume
\[ \Omega(\varepsilon) \propto \sqrt{\frac{\varepsilon}{\bar{\varepsilon}}} \]

partition function
Excited-State Quantum Phase Transitions (ESQPTs)

The phase-space criterion

\[ \rho(E) \propto \int \delta(E - H) \, d^f x \, d^f p = \frac{d}{dE} \int \Theta(E - H) \, d^f x \, d^f p \]

1D potential:

* Special type of ESQPT, in a sense stronger than 1st order. The same ESQPT results from a 1D inflection point with \( \frac{d}{dx} V = 0 \), see the inset**
“Cusp“ Hamiltonian – potential with the “cusp” type of catastrophe: basic 1D model for 1st & 2nd order QPTs

\[ \hat{H} = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + \frac{V(x)}{x^4 + ax^2 + bx} \]

\[ N \propto \frac{\sqrt{M}}{\hbar} \]

Textbook quantum mechanics!!!
Below we present numerical diagonalization for \( N = 10^2 \)
„Cusp“ Hamiltonian

1st order

\[ V \]

\[ b = -\frac{1}{4}, \quad b = 0, \quad b = +\frac{1}{4} \]

2nd order

\[ \psi_0^2 \]

\[ \psi_4^2 \]

\[ \pi = \pm \]

\[ \alpha = +1, \quad b=0 \]

\[ \alpha = -1 \]

\[ a = b = 0 \]
Generic ESQPT structures accompanying 1st & 2nd order QPTs

**1st order**
- \(b = -\frac{1}{4}\)
- \(b = 0\)
- \(b = +\frac{1}{4}\)

**2nd order**
- \(b = 0\)
- \(a = -1\)
- \(a = b = 0\)
- \(a = +1, b = 0\)
Examples

2D vibron model

$E = 0$

$H = (1 - \xi) n_b - \frac{\xi}{N} (Q \cdot Q)$

$Q = s^+ \tilde{b}_\lambda + \tilde{b}_\lambda^+ s$

ESQPT precursors scale with $l/N$
Examples

O(6) angular momentum $\nu = 0$

O(3) angular momentum $l = 0$

"continuous" ESQPT

$H = \frac{\eta}{N} n_d - \frac{1-\eta}{N^2} Q_0 \cdot Q_0$

$Q_0 = d^+ s + s^+ \tilde{d}$
Examples

$\psi^2$

$E$

$i = 15$

$i = 10$

$N=80$

sd-IBM
Examples

2-level fermion pairing model

Some references:
✓ Classical anomalies vs. singularities in quantal spectra:
  1D: Cary, Rusu... 1993, 2D (monodromy): Cushman, Bates 1997, Child .........
✓ Lipkin model: Heiss, Müller 2002, Leyvraz, Heiss... 2005,
✓ Pairing model: Reis, Terra Cunha, Oliviera, Nemes 2005
✓ IBM: Cejnar, Heinze, Macek, Jolie, Dobeš 2006
✓ 2D vibron & pairing models: Caprio, Cejnar, Iachello 2008
✓ 3D vibron model: Pérez-Bernal, Iachello 2008
✓ Cusp model: Cejnar, Stránský 2008
✓ Quantum optical models: Pérez-Fernández, Cejnar, Relaño, Dukelsky, Arias 2010
✓ Other studies being performed in the context of quantum optics, BECs etc.
“Flow of levels”: a semiclassical approach based on the old-QM quantization scheme
[Einstein-Brillouin-Keller, 1917-1926-1958] [Bohr-Sommerfeld-Wilson, 1915]

\[ H = \frac{1}{2m(x)} p^2 + V(x \mid \xi) \]

\[ V(x \mid \xi) \propto (1 - \xi)x^4 + \xi x^2 \]

position-dependent mass term appears in all boson models

Action along periodic orbit

\[ 2 \int \sqrt{2m(x)[E - V(x)]} dx = S(E) \]

\( q \in c.\text{domain} \)
“Flow of levels”: a semiclassical approach based on the old-QM quantization scheme

[Einsein-Brillouin-Keller, 1917-1926-1958]
[Bohr-Sommerfeld-Wilson, 1915]

\[ S_n = \int p(x) \, dx = \left( n + \frac{1}{4} \right) \left( \frac{2\pi \hbar}{\hbar} \right) \]

Semiclassical energy curves ⇒ contours of the dependence \( S(E,\xi) \)

Action along periodic orbit

\[ 2 \int \sqrt{2m(x)[E - V(x)]} \, dx = S(E) \]

\( q \in \text{c.domain} \)

\( E = +0.1 \)

\( E = 0 \)

\( E = -0.1 \)
2D vibron model

\( l = 0 \)
Divergent level density and curvature of level energy.

Very roughly:
\[ \frac{\partial^2 E_i}{\partial \xi^2} \approx \frac{\text{const}}{\log |E|} \]

Sharper than:
\[ \propto |E|^k \quad k \in (0,1) \]
Excited-State Quantum Phase Transitions (ESQPTs)

The phase-space criterion

\[ \rho(E) \propto \int \delta(E - H) \, df_x \, df_p = \frac{d}{dE} \int \Theta(E - H) \, df_x \, df_p \]

\[ \Gamma(E) \]

\[ \Omega(E) \]

2D potential:

- **local minimum/maximum**
  - continuous (2nd order)

- **saddle point**
  - continuous (no order)*

* Precursors difficult to distinguish from the 1st order
Excited-State Quantum Phase Transitions (ESQPTs)

The phase-space criterion

$$\rho(E) \propto \int \delta(E - H) \, d^f x \, d^f p = \frac{d}{dE} \left( \int \Theta(E - H) \, d^f x \, d^f p \right)$$

**2D potential:**

- local minimum/maximum
  - continuous (2\textsuperscript{nd} order)

- saddle point
  - continuous (no order)*

* Precursors difficult to distinguish from the 1\textsuperscript{st} order

**Question of chaos:** ESQPTs rely on structural changes induced by close approach of levels. However, generic multi-dimensional systems are chaotic and thus exhibit repulsion of levels! Therefore, further weakening of ESQPT signatures is expected.
2D collective model  (analog of 5D nuclear collective model for $l=0$)

\[ H = -\frac{1}{2N^2} \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] + r^4 + Ar^2 + Br^3 \cos 3\varphi \]

\[ \propto \frac{\sqrt{M}}{\hbar} \]
2D collective model (analog of 5D nuclear collective model for $l=0$)

$$H = -\frac{1}{2\lambda^2} \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] + r^4 + Ar^2 + Br^3 \cos 3\varphi$$

$$\propto \frac{\sqrt{M}}{\hbar}$$
2D collective model  (analog of 5D nuclear collective model for $l=0$)

$$H = -\frac{1}{2\Sigma^2} \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right] + r^4 + Ar^2 + Br^3 \cos 3\phi$$

$$\propto \frac{\sqrt{M}}{\hbar}$$
2D collective model

\[ B = 1, \mathcal{N} = 10^3 \]
$B = 1, \mathcal{N} = 10^3$

level density

$\mathcal{N} \in [2 \cdot 10^2, 10^4]$
level density

\[ N \in [2 \cdot 10^2, 10^4] \]

\[ B = 1, N = 2 \cdot 10^2 \]

continuous 2nd order
\[ H = \eta n_d - \frac{1 - \eta}{N} Q_{\chi_0} \cdot Q_{\chi_0} \]

\[ Q_{\tilde{\chi}} = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)} \]

\[ \chi_0 = -\frac{\sqrt{2}}{2} \]

**ESQPTs**

Continuous & 2\textsuperscript{nd} order

**QPT**

1\textsuperscript{st} order

**phase-coexistence structures invisible** for moderate \( N \) (because of zero-point motion)
\[ H = \eta n_d - \frac{1-\eta}{N} Q_{\chi_0} \cdot Q_{\chi_0} \quad \chi_0 = -\frac{\sqrt{\eta}}{2} \]

\[ Q_{\chi} = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)} \]

\[ \eta = \frac{1}{2} \]

\[ N = 60 \]

\[ \rho \]

A... saddle point
B... local maximum
C... asymptotic value

ESQPTs
continuous & 2nd order
\[ H = \eta n_d - \frac{1-\eta}{N} Q_x \cdot Q_{x_0} \]

\[ Q_x = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)} \]

\[ \chi = -\frac{\sqrt{17}}{2} \]

\[ N = 40 \]

\[ l = 0 \]

\[ \eta = 0 \]

\[ N = 60 \]

\[ \rho \]

\[ A = C \]

ESQPTs
continuous & 2\textsuperscript{nd} order

\[ A = C \ldots \text{saddle point & asymptotic value} \]
\[ H = \eta n_d - \frac{1-\eta}{N} Q_{\chi_0} \cdot Q_{\chi_0} \quad \chi_0 = -\frac{\sqrt{7}}{2} \]

\[ Q_{\chi} = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)} \]

\[ \mathcal{E} = -\frac{1}{2} \left[ \left( \frac{\lambda}{N} \right)^2 + \left( \frac{\mu}{N} \right)^2 + \frac{\lambda}{N} \frac{\mu}{N} + \frac{3}{N} \frac{\lambda}{N} + \frac{3}{N} \frac{\mu}{N} \right] \]

\[ (\lambda, \mu) = (2N,0), (2N-4,2), (2N-8,4).... \]
\[ (2N-6,0), (2N-10,2), (2N-14,4).... \]
\[ (2N-12,0), (2N-16,2), (2N-18,4).... \]

\[ N=20 \quad L=0 \text{ states} \]

\[ N = 60 \]

\[ \rho \propto \frac{dS}{d\mathcal{E}} \]

\[ \eta = 0 \]

\[ A=C \quad \text{saddle point & asymptotic value} \]
\[
H = \eta n_d - \frac{1 - \eta}{N} Q_{x_0} \cdot Q_{x_0} \quad x_0 = -\frac{i}{2}
\]

\[
Q_x = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)}
\]

\[
\mathcal{E} = -\frac{1}{2} \left[ \left( \frac{\lambda}{N} \right)^2 + \left( \frac{\mu}{N} \right)^2 + \frac{\lambda}{N} \frac{\mu}{N} + \frac{3}{N} \frac{\lambda}{N} + \frac{3}{N} \frac{\mu}{N} \right]
\]

\[
(\lambda, \mu) = (2N, 0), (2N - 4, 2), (2N - 8, 4), \ldots
\]

\[
(2N - 6, 0), (2N - 10, 2), (2N - 14, 4), \ldots
\]

\[
(2N - 12, 0), (2N - 16, 2), (2N - 18, 4), \ldots
\]

\[
N=20 \\
L=0 \text{ states}
\]
Dicke model
Lecture I
Quantum Phase Transitions – singularities in quantal spectra affecting the ground state & excited states

Some memos:
• In “finite models”, infinite-size = classical!
• Ground-State QPTs as changes of the potential minimum
• Any singularity in the phase space above the minimum results in an Excited-State QPT
• ESQPTs are just a “microcanonical reformulation” of thermal phase transitions!
• Signatures of ESQPTs weaken with dimension (question of chaos)
• ESQPTs have dramatic (or less dramatic) dynamical consequences!