Understanding Chaos via nuclei

Pavel Cejnar

Institute of Particle and Nuclear Physics
Faculty of Mathematics and Physics
Charles University, Prague, Czech Republic

pavel.cejnar@mff.cuni.cz

with essential contribution of:

Pavel Stránský (now Mexico)
Michal Macek (now Jerusalem)

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According to Empedocles (cca.490-430 BC), the real world (*Cosmos*) originates in an interference of the exquisite world of perfect order (*Sphairos*) and the world of complete disorder (*Chaos*).
Understanding Chaos via nuclei

Outline of the Course

1) Chaos in Classical Physics
2) Chaos in Quantum Physics
3) Chaos in Nuclei
4) Chaos in the Geometric Model
5) Chaos in the Interacting Boson Model

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a) Poincaré
   - integrability vs. chaos
   - visualization of chaos
b) Lyapunov ...
   - methods to detect instability
c) Kolmogorov, Arnold, Moser
   - perturbed systems, KAM & Poincaré-Birkhoff theorems
d) Birkhoff, von Neumann
   - ergodicity, statistics...
In 1885, King Oscar II of Sweden decides to celebrate his 60th birthday by awarding a prize (gold medal and 2,500 golden crowns) for an important discovery in mathematical analysis.

One of the topics proposed is the $n$-body problem in celestial mechanics.

In 1888, Henri Poincaré submits a work entitled "Sur le problème des trois corps et les équations de la dynamique". The committee (Weierstrass, Hermite, Mittag-Leffler) awards him the price.

When the work is printed (160 pages), the editor complains about the lack of clarity of some parts. Poincaré finds an essential error. In 1890, he submits a new work (270 pages), which is already printed at his own expenses (> 2500 crowns; b.t.w., also the gold medal was later stolen from him).

The new work opens a golden mine for new research. In one of the later writings Poincaré explains: "It may happen that small differences in the initial conditions produce great ones in the final phenomena."
3-body problem

Example calculation from PhD Thesis of Pavel Stránský

$$m_1 = m_2 = 1$$
$$m_3 = 0.2$$

⇒ Chaos
**Integrable systems**

**Definition:** System with $f$ degrees of freedom is integrable if:

- $\exists$ $f$ integrals of motions: $\{F_i, H\} = 0 \quad i=1\ldots f$
- These integrals are all in involution: $\{F_i, F_j\} = 0$
- All the gradients $\nabla F_i$ (2$f$-dimensional) are linearly independent

**Properties:**
- Hamiltonian can be expressed as: $H(\vec{p}, \vec{q}) \equiv H(F_1, F_2, \ldots, F_f)$
- $\exists$ canonical transformation to the action-angle variables
  $\Rightarrow$ the trajectories in the phase space confined on $2f$-dimensional tori

$$\begin{align*}
\vec{p} \rightarrow & \left\{ \begin{array}{l}
I_i = \text{const} \\
\vec{q} \rightarrow & \left\{ \begin{array}{l}
\theta_i = \omega_i t + \omega_{i0}
\end{array} \right. \end{array} \right.
\end{align*}$$

- the system is "nice & solvable" though not always fully analytic:

\[
\{A, B\} = \sum_i \left( \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial q_i} \right)
\]
Non-integrable systems

\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \]

Trajectories \[ \rightarrow \] flow in the phase space

- incompressible fluid
  \[ \Rightarrow \text{volume conserved (Lioville theorem } \frac{d}{dt} \rho = 0 \text{) } \]
  \[ \text{shape} \text{ can become arbitrarily complicated} \]

\[ \Rightarrow \text{complex structures in the phase space} \]
\[ \Rightarrow \text{exponential sensitivity to initial conditions} \]
\[ \Rightarrow \text{the system is not solvable} \quad (\text{but some think it is still nice}) \]

\[ \Rightarrow \text{Chaos} \]
### Poincaré map

Visualization of dynamics for systems with $f=2$

Section of the phase space
Dots = passages of one/more trajectories through the section

- **Order** =► dots form patterns (topological circles)
- **Chaos** =► dots make “mash” (random filling)
- **Order + Chaos** =► the dots produce patterns + mash

---

**4D Phase Space**

Coordinate$_2$ Fixed
Momentum$_2$ determined from energy $E$
⇒ each point in the section plane is crossed by just 1 trajectory
Billiard systems

2D cavities of different shapes (integrable or nonintegrable)
Poincaré sections do not depend on energy $E$

Eccentric annular billiard

Bohigas et al. NPA 560, 197 (1993)
Frischat, Doron PRE 57, 1421 (1998)
Dembowski et al. PRL 84, 867 (2000)
Nuclear collective models
Geometric Model & Interacting Boson Model

In systems with a “soft” potential the Poincaré sections (orbit types) depend on energy $E$.

Bohr shape variables of the nucleus
**Interacting Boson Model**

**Integrable case:** Mexican hat potential $V(\beta)$
Energy set to the central maximum of the potential

**Non-integrable case:** Potential $V(\beta,\gamma)$ with 3 minima
Energy above the central maximum of the potential


M. Macek, P. Stránský, P. Cejnar, S. Heinze, J. Jolie, PRC 75 (2007) 064318

50,000 passages of 52 randomly chosen trajectories through the section $y=0$

Energy $E$ corresponds to the local maximum of the potential $V(\beta,\gamma)$
Geometric Model

Comparison of Poincaré maps on planes $x=0$ and $y=0$

100 trajectories $\times$ 1000 passages

Source: PhD thesis of P. Stránský
Geometric Model

P. Cejnar, P. Stránský, PRL 93 (2004) 102502
P. Stránský, M. Kurian, P. Cejnar, PRC 74 (2006) 014306
Lyapunov 1892

Dissertation “Общая задача об устойчивости движения”
later translated: "Sur le probleme general de la stabilite du mouvement"
A general task about the stability of motion

Linearized equation for the phase-space deviation
\[
\frac{d}{dt} \delta X_\alpha \approx \sum_\beta J_{\alpha\beta} \frac{\partial^2 H}{\partial X_\beta \partial X_\alpha} \delta X_\beta
\]

\[\delta \vec{X} = \begin{pmatrix} \delta q \\ \delta p \end{pmatrix}\]

\[J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\]

Leading Lyapunov Vector is apparent after sufficient time evolution

Uniform perturbations deform into an ellipse as the system evolves

\[
\text{Lyapunov exponent: } \sigma = \lim_{t \to \infty} \max_{\delta X(0)} \frac{1}{t} \ln \left| \frac{\delta \vec{X}(t)}{\delta \vec{X}(0)} \right|
\]

for unbounded systems

\[
\max \delta \vec{X}(t) \approx \delta \vec{X}(0) e^{\sigma t}
\]

\[\sigma = 0 \implies \text{order} \]
\[\sigma > 0 \implies \text{chaos} \]
Lorenz  circa 1960

Rediscovery of instability of motions in early computer simulations of a simple 12-variable weather model: start of the new era of chaos

Figure 1: Lorenz’s experiment: the difference between the start of these curves is only .000127. (Ian Stewart, Does God Play Dice? The Mathematics of Chaos, pg. 141)

Werner Heisenberg
"When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first."
Alignment method 2001

For bounded phase-space domains the long-time deviations of trajectories always saturate
⇒ Lyapunov exponents must be estimated from finite time intervals
⇒ computational problems

A more suitable method for probing the stability of individual trajectories was proposed in:
Skokos, JPA 34, 10029 (2001),
Skokos, Antonopoulos, Bountis, Vrahatis JPA 37, 6269 (2004)

Idea: For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximal Lyapunov exponent

Procedure:
• take 2 independent deviations and calculate evolution of their directions
• calculate parallel & antiparallel “alignment indices” $d_{\pm}(t) = ||\hat{\delta}_1(t) \pm \hat{\delta}_2(t)||$
• take the smaller index: $\text{SALI}(t) \equiv d_{\pm}(t) = \min\{d_{+}(t), d_{-}(t)\} \in [0, \sqrt{2}]$

$\hat{\delta}_1(t) \parallel \hat{\delta}_2(t)$ $\quad \hat{\delta}_1(t) \perp \hat{\delta}_2(t)$
[ chaotic … regular … ]
Each individual trajectory can be attributed by adjective “regular” or “chaotic”
Each individual trajectory can be attributed by adjective "regular" or "chaotic"
Regular phase-space fraction

\[ f_{\text{reg}}(E) = \frac{\Omega_{\text{reg}}(E)}{\Omega_{\text{tot}}(E)} \equiv \frac{\text{"Surface" of the given energy shell in the phase space occupied by regular orbits}}{\text{Total "surface" of the given energy shell in the phase space}} \]

\[ \Omega_{\text{tot}}(E) \equiv \int \delta(E - H(\bar{p}, \bar{q})) d\bar{p} d\bar{q} \]

The regular fraction can be estimated from:

1) Fraction of regular trajectories in a generated sample

\[ f'_{\text{reg}} = \frac{N_{\text{reg}}}{N_{\text{tot}}} \approx \]

2) Fraction of regular area in selected Poincaré section

\[ f''_{\text{reg}} = \frac{S_{\text{reg}}}{S_{\text{tot}}} \in [0,1] \]

[fully chaotic... transitional... ...fully regular]
Kolmogorov-Arnold-Moser (KAM)

Perturbation of an integrable system

What happens with regular tori?

a) they all just deform 😊
b) they all disappear 😞
c) some deform, some disappear

Which deform?
Which disappear?

KAM theorem ~

Consider an $f=2$ torus characterized by frequencies $\omega_1$ & $\omega_2$ and their ratio $\frac{\omega_2}{\omega_1} \equiv \mu$

Condition for the torus survival can be written as:

$$\left| \mu - \frac{m_1}{m_2} \right| > \frac{\text{const}}{|m_2|^{2+\epsilon}} \quad \forall m_1, m_2 = 1, 2, \ldots$$

This is a measure of irrationality of the number $\mu$
The “more irrational” the number $\mu$ is, the longer the orbit survives Tori close to rational (periodic) ones die first...
**Kolmogorov-Arnold-Moser (KAM)**

**Perturbation of an Integrable System**

What happens with regular tori?

- a) they all just deform
- b) they all disappear
- c) some deform, some disappear

Which deform? Which disappear?

**KAM theorem**

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Condition for the torus survival can be written as:

$$\left| \frac{\mu - \frac{m_1}{m_2}}{m_2} \right| > \frac{\text{const}}{|m_2|^{2+\varepsilon}} \quad \forall m_1, m_2 = 1, 2, \ldots$$

This is a **measure of irrationality** of the number $\mu$. The “more irrational” the number $\mu$ is, the longer the orbit survives. Tori close to rational (periodic) ones die first...
Statement:
“A deceased rational torus leaves an even number of periodic orbits”
\(\frac{1}{2}\) of them stable, \(\frac{1}{2}\) of them unstable

**Integrable System**
- Irrational tori
- Rational torus and a periodic orbit living on it (every point on the oval is the fixed point after \(n\) turns)

**Perturbed System**
- Survived irrational tori (deformed)
- Remnants of the rational torus:
  - Stable periodic orbits and the surrounding islands of regularity
  - Unstable periodic orbits
  \(\Rightarrow\) an even number of fixed points (stable & unstable) after \(n\) turns

Nageswaran Rajendran
http://chaos.physik.uni-dortmund.de/~eswar/PencilSketches.html
Nuclear examples:
[Macek, Stránský, Cejnar 2004-6]

⇒ Departure from integrability & proliferation of chaos is a rather complex & beautiful performance!
What to do in the middle of chaos?

survival guide

Do not panic!
Enjoy the positive features of chaos...

(Quasi) Ergodicity
[Birkhoff 1931, von Neumann 1932]

Only for chaotic systems!
• Any trajectory passes in any vicinity of any point on the given energy surface in the phase space
• Time averages can be replaced by phase-space averages

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T dt O(\bar{p}(t), \bar{q}(t)) \equiv \langle O \rangle_{\text{time}} \cong \langle O \rangle_{\text{phasespace}} \equiv \frac{1}{\Omega_{\text{tot}}(E)} \int_0^T d\bar{p} d\bar{q} \delta(E - H(\bar{p}, \bar{q})) O(\bar{p}, \bar{q}) \]

⇒ statistical predictability
What to do in the middle of chaos?

survival guide

Do not panic!
... and await the next round of order!

Geometric Model

Energy

Parameters

full integrability

partial regularity
quasisymmetries...

just a small increase of energy
What to do in the middle of chaos?

survival guide

Do not panic!
…and await the next round of order!

full integrability
partial regularity
quasisymmetries…

Geometric Model
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2) **Chaos in Quantum Physics**
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- **Einstein, Chirikov, Berry** et al.
  puzzle of quantum chaos
  classical-quantum correspondence
- **Wigner**
  random Hamiltonians
- **Bohigas** et al.
  conjecture for quantum chaos
- **Peres**
  quantum instability
  visualization of quantum chaos
Puzzle of Quantum Chaos

• **Einstein 1917**

  Question on quantization of chaotic systems

  Bohr-Sommerfeld rule in integrable systems:

  \[
  \oint_{C_i} \mathbf{p} \cdot d\mathbf{q} = n_i \hbar \quad n_i = 1,2,3,\ldots
  \]

  \[
  i = 1\ldots f
  \]

  How about if there are no tori?

• **Heisenberg, Schrödinger, von Neumann, Dirac... 1925-27**

  Autonomous formulation of quantum mechanics
  ⇒ the problem of chaos disappeared

• **Casati, Chirikov, Berry, Ford, Izrailev... 1970’s**

  How to define chaos in quantum systems?
  1) The problem of linearity
  2) The problem of long-time evolution
Puzzle of Quantum Chaos

**Linearity of QM**

\[ |\Psi(t)\rangle = e^{-i\frac{\hat{H}_t}{\hbar}} |\Psi(0)\rangle \]

\[ |\Psi'(t)\rangle = e^{-i\frac{\hat{H}_t}{\hbar}} |\Psi'(0)\rangle = \left[\sqrt{1-\delta^2} |\Psi(t)\rangle + \delta |\Psi_\perp(t)\rangle\right] \]

⇒ the distance of solutions remains the same for all times:

\[ ||\Psi'(t) - |\Psi(t)\rangle|| = \sqrt{\langle\Psi'(t) - \Psi(t) | \Psi'(t) - \Psi(t)\rangle} \]

\[ = \sqrt{(\sqrt{1-\delta^2} - 1)^2 + \delta^2} \approx \delta \]

⇒ no butterfly effect in the Hilbert space

**Long-time evolution of a classical chaotic system**

Evolution in the phase space

at this stage quantum fluctuations start playing an important role => classical description no more applicable

⇒ quantum suppression of chaos
Hyperion

Saturn’s satellite known by its potato shape and chaotic rotation

\[ \sigma^{-1} \equiv \tau_{\text{chaos}} \approx 100 \text{ days} \]

[ M. Berry 2001]

Quantum mechanics predicts that long-time dynamics of Hyperion cannot be chaotic. The time scale for the quantum suppression of chaos is:

\[ \tau_{\text{suppression}} \approx 37 \text{ years} \]

Solution of this paradox: \textit{decoherence} (e.g., due to interactions with solar photons)

\[ \tau_{\text{decoherence}} \approx 10^{-53} \text{ s} \]
Role of Decoherence

Interaction of the quantum system with environment helps to restore the classical-quantum correspondence.

Example:
Greenbaum, Habib, Shizume, Sundaram, PRE 76, 046215 (2007)
A study of chaotic Duffing oscillator in contact with external environment. The strength of system-environment interaction is varied by changing the diffusion coefficient $D$.

Wigner quasiprobability distribution (quantum)

$W(x, p) = \frac{1}{2\pi\hbar} \int \rho(x - \frac{p}{2}, x + \frac{p}{2}) e^{i\hbar p y} dy$

Wigner distribution

$\rho_{\text{cl}}(p, x)$ classical probability distribution

Phase-space distributions
Abstract: There is no quantum chaos in the sense of exponential sensitivity to initial conditions, but there are several novel quantum phenomena which reflect the presence of classical chaos. The study of these phenomena is quantum chaology.

List of topics (incomplete)

- correlation properties of discrete energy spectra of bound states
- morphologies of wavefunctions of bound states
- long-time evolution of nonintegrable quantum systems
- sensitivity of quantum dynamics to Hamiltonian perturbations
- patterns of quantum expectation values in energy eigenstates
Compound nucleus
Niels Bohr 1936

Resonances in the neutron-nucleus cross sections:

\[ \Gamma \approx 10^{\pm 100} \text{ meV} \Leftrightarrow \tau \approx 10^{-13} \div 10^{-14} \text{ s} \]

Coceva, Stefanon, NPA (1979)

Landon, PR (1955)

Bohr
Nature (1936)
Random Hamiltonian \[ \text{Wigner 1955} \]

How to describe neutron resonances?
No chance of detailed theory! Only a statistical approach possible.

Take the Hamiltonian at random!

\[
\hat{H} = \begin{pmatrix}
H_{11} & H_{12} & \cdots & H_{1n} \\
H_{21} & H_{22} & \cdots & H_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n1} & H_{n2} & \cdots & H_{nn}
\end{pmatrix}
\]

Hermitian

\[
H_{ij} = H_{ji} \in \mathbb{R} \quad \text{...invariance}
\]

\[
H_{ij} = H_{ji}^* \in \mathbb{C} \quad \text{...noninvariance}
\]

under time-reversal

Metric and measure in the space of matrices:

\[
d s^2 = \text{Tr}[\delta \hat{H} \delta \hat{H}^+] = \sum_i \delta H_{ii}^2 + 2 \sum_{i>j} \left| \delta H_{ij} \right|^2
\]

\[
d\hat{H} = \sqrt{\text{det } g_{\mu\nu}} \prod_i dH_{ii} \prod_{i>j} d\text{Re } H_{ij} \prod_{i>j} d\text{Im } H_{ij}
\]

invariant under unitary/orthonormal transformations

Task: Find distribution \( W(\hat{H}) \) in the space of matrices such that the probability \( W(\hat{H})d\hat{H} \) is invariant all unitary or orthonormal transformations \( \hat{U} \) or \( \hat{O} \) (for simplicity assume statistical independence of matrix elements)

Solution: Gaussian Unitary/Orthogonal Ensemble = GUE / GOE

\[
W(\hat{H}) = N \exp\left[ -b \text{Tr } \hat{H}^2 \right] = N \exp\left[ -b \left( \sum_i H_{ii}^2 + \sum_{i>j} (\text{Re } H_{ij})^2 + \sum_{i>j} (\text{Im } H_{ij})^2 \right) \right]
\]

This distribution maximizes the entropy \( S = -\int W(\hat{H}) \ln W(\hat{H}) d\hat{H} \) under constraint \( \int \text{Tr } \hat{H}^2 W(\hat{H}) d\hat{H} = \text{const} \)
Spectral correlations

We look for the statistical distribution of the set of eigenvalues
Diagonalization transformation depending on some “angles”

\[ \hat{H} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix} \rightarrow \hat{U} \text{ or } \hat{O} \rightarrow \begin{pmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_n \end{pmatrix} \]

- Repulsive Coulomb potential in 2D (force \( \sim 1/r \))
- Oscillator trap

\[
\begin{align*}
P(E_1, E_2, \ldots, E_n) &= N \prod_{i > j} |E_i - E_j|^{\beta} \exp \left[ -b \sum_i E_i^2 \right] \\
&\propto \exp \left[ -b \sum_i E_i^2 - \beta \sum_{i > j} \ln |E_i - E_j| \right]
\end{align*}
\]

Thermal distribution of equally charged 2D particles in an oscillator trap at inverse temperature:

\[
\beta = \begin{cases} 
1 & \text{GOE} \\
2 & \text{GUE} 
\end{cases}
\]

\( \Rightarrow \) repulsion of neighboring levels
\( \Rightarrow \) correlations mediated by Coulomb interactions

Contrast: particles with no charges – Poisson distribution with no correlations
Spectral correlations

Wigner 1955

To quantify the correlations, one has to remove the variations of the spectrum due to changes of the mean level density. This procedure is called "unfolding".

M. Hanke (2006): spectrum of a 2D system with a quartic type of potential

\[ N(E) = \int_{-\infty}^{E} \sum_{i} \delta(E' - E_i) dE' \]

\[ E \rightarrow \bar{E} = \int_{-\infty}^{E} \tilde{\rho}(E') dE' \]

\[ \rho(\lambda) \propto \sqrt{\frac{\beta}{\lambda^2} - E^2} \]

Fluctuations around the uniform staircase carry information on the correlation properties.

Various fluctuation measures are available and employed in literature.
Spectral correlations

Wigner 1955

1) Short range:
Nearest Neighbor Spacing

\[ P_{\text{NNS}}(s) \propto s^\beta \]

Strong level repulsion

\[ \langle s \rangle = 1 \]

\[ s = \overline{E}_i - \overline{E}_{i-1} \]
Spectral correlations  Wigner 1955

2) Long range:
spectral rigidity, number variance, noise analysis...

Example: **Number variance**

\[ \Sigma^2(L) = \left\langle N(\bar{E}, \bar{E} + L)^2 \right\rangle - \left\langle N(\bar{E}, \bar{E} + L) \right\rangle^2 \]

number of levels in energy interval of length \( L \) in the unfolded spectrum

averaging over the spectrum

**Poisson** distribution of levels (no correlations)

\[ \Sigma^2(L) = L \]

**GOE**

\[ \Sigma^2(L) \approx \frac{1}{\pi^2} \left[ \ln(2\pi L) + \gamma + 1 \right] \]

very rigid spectrum
Universality

\[ W(\hat{H}) = N \exp\left[-b \text{Tr} \hat{H}^2\right] = N \exp\left[-b \left( \sum_i H_{ii}^2 + \sum_{i>j} (\text{Re} H_{ij})^2 + \sum_{i>j} (\text{Im} H_{ij})^2 \right)\right] \]

This distribution **maximizes the entropy** under constraint \( \int \text{Tr} \hat{H}^2 W(\hat{H}) d\hat{H} = \text{const} \)

\[ S = -\int W(\hat{H}) \ln W(\hat{H}) d\hat{H} \]

Robust predictions applicable in:

- physics
- engineering
- economy
- medicine
- mathematics
- …………

Philosophical reflections:

**Riemann hypothesis:**
All points $\zeta(s) = 0$ in the complex plane of $s$, except of “trivial zeros” $s = -2, -4, -6, \ldots$, are located at line $s = \frac{1}{2} + iy$

Important consequences for number theory and many other branches of mathematics

Numerical results up to $N_{\text{zero}} \approx 10^{20}$ consistent with GUE. Is zeta function related to quantum chaos?

---

B. Cipra: A prime case of chaos (AMS, 1999)
Bohigas conjecture 1984


**Statement:**
Spectra of classically chaotic systems follow the GOE/GUE statistics.

**Quantum examples:**
(a) spectrum of Sinai billiard
(b) spectrum of hydrogen atom in strong magnetic field
(c) spectrum of NO$_2$ molecule

**Non-quantum examples:**
(d) spectrum of acoustical resonances of Sinai-shaped quartz block
(e) microwave spectrum of a 3D chaotic cavity
(f) vibration spectrum of a quarter-stadium shaped plate

Spectra of integrable systems

\( f \) commuting integrals of motions
with \( f = \) number of quantum degrees of freedom

\[ [\hat{H}, \hat{C}_i] = 0 = [\hat{C}_i, \hat{C}_j] \quad i, j = 1 \cdots f \]

\( \Rightarrow \) \( \exists \) simultaneous eigenbasis \( \{c_1, c_2, \cdots, c_f\} \)

which also diagonalizes the Hamiltonian \( \hat{H} = \)

\( \langle c_1 c_2 \cdots c_f | \hat{H} | c'_1 c'_2 \cdots c'_f \rangle = h_{c_1 c_2 \cdots c_f} \delta_{c_1' c_1} \delta_{c_2' c_2} \cdots \delta_{c_f' c_f} \)

\( \Rightarrow \) eigenvalues are totally **uncorrelated**

(after the unfolding)

**Poisson statistics**

Number of levels in an interval \( \Delta \bar{E} \) of the *unfolded* spectrum:

\[ P_{\Delta \bar{E}}(N) = \frac{\left(\Delta \bar{E}\right)^N}{N!} e^{-\Delta \bar{E}} \Rightarrow \text{Nearest Neighbor Spacing distribution} \quad (N=0) \]

\[ P_{\text{NNS}} = e^{-s} \]

Usually valid in integrable (fully regular) systems.

**Numerous exceptions** known, e.g., in the SU(3) limit of the interacting boson model...
Billiards are popular objects for testing the Bohigas conjecture. **Specific feature:** classical motions do not depend on energy.
Between order & chaos

Poisson ($\omega=0$)

\[ P_{\text{NNS}} = e^{-s} \]

Wigner ($\omega=1$)

\[ \frac{N_\omega}{(\omega+1)\alpha_\omega} s^\omega e^{-\alpha_\omega s^{\omega+1}} \]

\[ \alpha_\omega = \Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} \]

Brody distribution (1973)

\[ P_{\text{NNS}} = \frac{\pi}{2} e^{-\frac{\pi}{4}s^2} \]

**Example:** fit of data from a narrow energy interval (geometric model)

Picture from PhD Thesis of Pavel Stránský
**Between order & chaos**

Brody distribution

\[ P_{\text{NNS}} = e^{-s} \quad \text{for} \quad \omega = 0 \]

\[ P_{\text{NNS}} = N_\omega s^\omega e^{-\alpha_\omega s^{\omega+1}} \quad \text{for} \quad \omega = 1 \]

\[ \alpha_\omega = \Gamma \left( \frac{\omega+2}{\omega+1} \right)^{\omega+1} \]

**Linear fitting method**

Prosen, Robnik, JPA 26,2371(1993)

Example: fit of data from a narrow energy interval (geometric model)

- **Poisson** \( \omega = 0 \)
- **Wigner surmise (GOE)** \( \omega = 1 \)
Between order & chaos

Brody distribution
(1973)

\[ P_{\text{NNS}} = e^{-s} \]

\[ N_\omega = (\omega+1) \alpha_\omega \]

\[ P_{\text{NNS}} = N_\omega s^\omega e^{-\alpha_\omega s^{\omega+1}} \]

\[ \alpha_\omega = \Gamma(\frac{\omega+2}{\omega+1})^{\omega+1} \]

\[ P_{\text{NNS}} = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} \]

**Example:** fit of data from a narrow energy interval (geometric model)
Fidelity [Peres, PRA 30, 1610 (1984)]
Quantum chaotic systems show an “exponential sensitivity” to details of the Hamiltonian (to its control parameters)

\[ F(t) = \left| \left< e^{-i\frac{\hat{H}t}{\hbar}} \Psi_0 | e^{-i\frac{(\hat{H}+\delta\hat{H})t}{\hbar}} \Psi_0 \right> \right|^2 \]

References:
Goussev, Jalabert, Pastewski, Wisniacki, Scholarpedia, 7(8):11687 (2012)

Perturbative expansion:
\[ F(t) \approx \sum_i p_i^2 + 2 \sum_{i>j} p_ip_j \cos \left( \frac{(\delta E_i - \delta E_j)t}{\hbar} \right) \]
with \( p_i = \left| \langle E_i | \Psi_0 \rangle \right|^2 \)

**Quadratic regime**
\[ F(\delta t) \approx 1 - 2 \frac{\langle E^2 \rangle_{\psi_0} - \langle E \rangle_{\psi_0}^2}{\hbar^2} \delta t^2 \]

**Gaussian/exponential regime**
For weak perturbations.
For strong perturbations. For “global perturbations” the decay constant saturates at the Lyapunov exponent.

Figures from perturbation strength classical Lyapunov exp.
Spectral lattices [Peres, PRL 53, 1711 (1984)]
A visual method to detect chaos in systems with $f=2$
Quantum analog of Poincaré maps in classical systems

Integrals of motions for non-integrable systems:

$$
\hat{P} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{P}_H(t) dt
$$

Time average of an arbitrary observable $P$

$$
[\hat{P}, \hat{H}] = 0 \quad \bar{P}_i \equiv \langle \psi_i | \hat{P} | \psi_i \rangle = \langle \psi_i | \hat{P} | \psi_i \rangle = \langle P \rangle_i
$$

for stationary states

Energy spectrum as "Peres lattice"

Regular system (integrable)
\Rightarrow ordered lattice
(the new motion integral must be a function of old ones)

Mixed system (regular & chaotic)
\Rightarrow mixed ordered & disordered lattice

Examples from Geometric Model

Chaotic system \Rightarrow disordered lattice
(Contracted to narrow band due to ergodicity)
Understanding Chaos via Nuclei

Outline of the Course

1) Chaos in Classical Physics
2) Chaos in Quantum Physics
3) Chaos in Nuclei
4) Chaos in the Geometric Model
5) Chaos in the Interacting Boson Model

a) Data ensembles
   level statistics at high & low energies
   statistical paradigm

b) Model predictions
   many-body models
   mean field models
   collective models

Fiera 2013
Evidence for the applicability of random-matrix theory to nuclear spectra is reviewed. In analogy to systems with few degrees of freedom, one speaks of chaos (more accurately, quantum chaos) in nuclei whenever random-matrix predictions are fulfilled. An introduction into the basic concepts of random-matrix theory is followed by a survey over the extent experimental information on spectral fluctuations, including a discussion of the violation of a symmetry or invariance property. Chaos in nuclear models is discussed for the spherical shell model, for the deformed shell model, and for the interacting boson model. Evidence for chaos also comes from random-matrix ensembles patterned after the shell model such as the embedded two-body ensemble, the two-body random ensemble, and the constrained ensembles. All this evidence points to the fact that quantum chaos is a generic property of nuclear spectra, except for the ground-state regions of strongly deformed nuclei.

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I. Constrained ensembles

II. Summary and Conclusions

...the following is NOT such a review!
Spectral correlations – high energy

- Haq, Pandey, Bohigas, PRL 48,1086 (1982)
- Bohigas, Haq, Pandey, in Nuclear Data for Science and Technology (Dordrecht, Reidel, 1983), p. 809

Nuclear Data Ensemble

Neutron Resonances
complete sequences of typically ~150-170 resonances with the same spin & parity in one heavy nucleus

Proton Resonances
shorter (~60-80) complete sequences of the same spin & parity in one heavy nucleus

=> In total, 1726 spacings
Spectral correlations – low energy

For some isotope groups (and particularly for some spin-parity values), significant deviations from the Wigner distribution are observed.

von Egidy, Behkami, Schmidt (1986, 1988)
von Egidy, Bucurescu (2009)
Statistical approach

“Quasi-continuum” (= *Apeiron*)
Completely known part of the spectrum

In the “quasi-continuum” part of the spectrum, statistical assumptions are employed:
(1) Mean values calculated from nuclear models
   (level densities, strength functions)
(2) Fluctuations determined from random-matrix theory

(Repeated) Monte-Carlo realizations of energy levels & transition rates in “quasi-continuum”

Known levels & transition rates

Figure: courtesy of F. Bečvář

Bohr (1936)
Non-statistical effects

Koehler, Bečvář, Krtička, Harvey, Guber, PRL 105, 072502 (2010)

Distribution of eigenfunction components in an arbitrary basis: GOE for large dimension $n$ predicts a Gaussian:

$$P(\alpha) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{n}{2} \alpha^2}$$

⇒ transition amplitudes $A = \langle \Psi_f | \hat{T} | \Psi_i \rangle$ are also Gaussian and the transition probability $P = |A|^2$ is $\chi^2$ distributed. Application to reduced neutron widths $\Gamma_n^0 = \Gamma_n / \sqrt{E_{\alpha}}$ of neutron resonances [Porter, Thomas, PR 104, 483 (1956)]:

$$x = \frac{\Gamma_n^0}{\langle \Gamma_n^0 \rangle} \quad P_{PT}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2} \quad \chi^2 \text{ with } \nu = 1$$

degree of freedom

⇒ This looks like perfect agreement with Porter-Thomas prediction, but if the NDE uncertainties are taken into account, one gets a different picture!
Many-body chaos

Shell model

Many papers, review in:
Zelevinsky, Brown, Frazier, Horoi

Random two-body interactions

(Two-Body Random Ensembles)

Revival: Johnson, Bertsch, Dean, PRL 80, 2749 (1998)

• The 2-body interaction of particles is taken as a GOE Hamiltonian in 2-body space
• The resulting $N$-body Hamiltonian has many zero matrix elements
• Nevertheless, the $N$-body spectra show the GOE correlations

Calculation for 12 valence nucleons in the 2s1d shell (lower shells inert, upper shells do not exist)
**One-body chaos**

**Single-particle dynamics:** simulations of nucleon motions in a diffuse and/or deformed mean-field potential

- **Poincaré map**
- **quadrupole deformation,** “diffuse” potential
  - Arvieu, Brut, Carbonell, Touchard, Rozmej
  - PRA 35, 2389 (1987)
  - NPA 545, C497 (1992)

- **quadrupole + octupole + hexadecapole deformation,** “soft”/“hard” potential
  - Heiss, Nazmitdinov, Radu
  - PRL 72, 2351 (1994)
  - PRL 73, 1235 (1994)
  - PRC 52, 3032 (1995)
Collective chaos

**Interacting Boson Model** [Iachello, Arima 1975]

- **Paar, Vorkapic, Dieperink**
  PLB 205, 7 (1988); PRC 41, 2397 (1990), PRL 69, 2184 (1992)
- **Alhassid, Whelan, Novoselsky**
- **Mizusaki, Yoshinaga, Shigehara, Cheon**
  PLB 269, 6 (1991)
- **Canetta, Maino**
  PLB 483, 55 (2000)
- **Cejnar, Jolie, Macek, Casten, Dobeš, Stránský**
- **Macek, Leviatan**
  PRC 84, 0413302 (2011), PLB 714, 110 (2012)

**Geometric Model** [Bohr 1952]

- **Bolotin, Gonchar, Inopin, Berezovoj, Cherkaskiy**...
- **Cejnar, Stránský, Kurian, Hruška**

quadrupole vibrations

coming soon in this Course…
Understanding Chaos via nuclei

Outline of the Course

1) Chaos in Classical Physics
2) Chaos in Quantum Physics
3) Chaos in Nuclei
4) Chaos in the Geometric Model
5) Chaos in the Interacting Boson Model

- a) Model
  - general features
  - scaling properties
  - quantization schemes
- b) Chaos results
  - comparison of classical & quantum measures
  - Peres lattices

Fiera 2013
Based on papers:

- **Regular and chaotic vibrations of deformed nuclei with increasing gamma rigidity**
- **Classical chaos in the geometric collective model**
- **Order and chaos in the geometric collective model**
- **Quantum chaos in the nuclear collective model: Classical-quantum correspondence**
- **Quantum chaos in the nuclear collective model: II. Peres lattices**
- **Chaotic dynamics in collective models of nuclei**
- **Regular and chaotic collective modes in nuclei**
  P. Cejnar, P. Stránský, M. Macek, Nuclear Physics News 21 (2011) 22

Earlier work:

Bolotin, Gonchar, Inopin, Berezovoj, Cherkaskiy...
YF 42, 53 (1985); YF 45, 350 (1987);
Ph. At. Nucl. 58, 1499 (1995); Ph. Part. Nucl. 34, 194 (2003);
PLA 323, 218 (2004)
Geometric Model

\[ H = \sum_{i=x,y,z}^N \frac{J_i^2}{2M} + \frac{1}{2M} \left( \pi_\beta^2 + \frac{\pi_\gamma^2}{\beta^2} \right) + \ldots + A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 + \ldots \]

\[ V = A(x^2 + y^2) + B(x^3 - 3y^2x) + C(x^2 + y^2)^2 \]

\[ T_{\text{rot}} = \frac{1}{2M} \left( \pi_x^2 + \pi_y^2 \right) \]

\[ T_{\text{vib}} = \frac{1}{2M} \left( \pi_x^2 + \pi_y^2 \right) \]

\[ A \approx \sum_{i=x,y,z} \pi_i^2 \]

\[ \pi_i^2 \approx \frac{1}{2M} \left( \pi_x^2 + \pi_y^2 \right) \]

\[ H = \frac{\sqrt{5}}{2M} \left[ \pi_\times \times (0) + \ldots + \sqrt{5}A[\alpha \times \alpha]^{(0)} - \sqrt{\frac{35}{2}}B[\alpha \times \alpha]^{(2)} \times \alpha \right]^{(0)} + 5C[\alpha \times \alpha]^{(0)} \right]^2 + \ldots \]

Hamiltonian

Shape variables

\[ x \equiv \beta \cos \gamma = \alpha_0^{(2)} \bigg|_{\text{PAS}} \]

\[ y \equiv \beta \sin \gamma = \sqrt{2} \Re \alpha_{\pm 2}^{(2)} \bigg|_{\text{PAS}} \]

Angular momentum \( J_\mu = -i\sqrt{10}[\alpha \times \pi^*]_\mu^{(1)} = 0 \)

\( \Rightarrow \) effectively **2D system**

An appropriate coordinates are obtained by the parameterization of \( \alpha \) in the Principal Axes System (obtained by the diagonalization of \( \alpha_{ij} \)):

\[ \alpha_{\pm 1}^{(2)} \bigg|_{\text{PAS}} = 0 = \Im \alpha_{\pm 2}^{(2)} \bigg|_{\text{PAS}} \]

\[ \alpha_0^{(2)} \bigg|_{\text{PAS}} = 0 = \sqrt{2} \Re \alpha_{\pm 2}^{(2)} \bigg|_{\text{PAS}} \]
Geometric Model

\[ H = \sum_{i=x,y,z} \frac{J_i^2}{2T_{\text{rot}}} + \frac{1}{2M} \left( \pi_\beta^2 + \frac{\pi_\gamma^2}{\beta^2} \right) + \ldots + A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 + \ldots \]

\[ V = A(x^2 + y^2) + B(x^3 - 3y^2x) + \frac{C}{\alpha} (x^2 + y^2)^2 > 0 \]

Relation to the Hénon-Heiles system

Hénon, Heiles, Astron. J. 69, 73 (1964)

...paradigm of chaos!!! Model inspired by motions of stars in an axially symmetric galactic potential. The cylindrical variables substituted: \((z,r) \rightarrow (x,y)\).

The HH potential is a toy example demonstrating the complexity of the problem.

Shape variables

\[ x \equiv \beta \cos \gamma = \alpha_0^{(2)} \mid_{\text{PAS}} \]

\[ y \equiv \beta \sin \gamma = \sqrt{2} \text{Re} \alpha_{\pm 2}^{(2)} \mid_{\text{PAS}} \]
**Geometric Model**

\[
H = \sum_{i=x,y,z} J_i^2 + \frac{1}{2M} \left( \frac{\pi^2}{\beta^2} + \frac{\pi^2}{\gamma^2} \right) + \ldots + A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 + \ldots
\]

\[
V = A(x^2 + y^2) + B(x^3 - 3y^2x) + C (x^2 + y^2)^2 > 0
\]

---

**Equilibrium shape diagram**

For \( B=0 \) the system is integrable.

**Shape variables**

\[
x \equiv \beta \cos \gamma = \alpha_0^{(2)} \mid_{\text{PAS}}
\]

\[
y \equiv \beta \sin \gamma = \sqrt{2} \text{ Re} \alpha_{\pm 2}^{(2)} \mid_{\text{PAS}}
\]
Scaling invariance

\[ H = \sum_{\ell=x,y,z} \frac{J_\ell^2}{2\mathcal{T}_{\text{rot}}} + \frac{1}{2M} \left( \frac{\pi_\beta^2}{\beta^2} + \frac{\pi_\gamma^2}{\beta^2} \right) + \ldots + A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 + \ldots \]

\[ V = A(x^2 + y^2) + B(x^3 - 3y^2x) + C(x^2 + y^2)^2 > 0 \]

4 control parameters:
- \( M, A, B, C \)

3 independent scales:
- \( E, \beta, t \) (momenta)

\[ E = aE' \quad \beta = b\beta' \quad t = ct' \]

\[ A' = \frac{b^2}{a} A \quad B' = \frac{b^3}{a} B \quad C' = \frac{b^4}{a} C \quad M' = \frac{a^2}{b^2} M \]

\[
\frac{B'^2}{A'C'} = \frac{B^2}{AC} \equiv \tau
\]

scaling invariant

= 1 essential control parameter

+ classicality parameter

\[ \frac{\hbar^2}{M} \equiv \kappa \]

This path crosses all non-equivalent configurations

... relevant in the quantum case
(scales the density of quantum states)
Can be replaced by a "position-dependent" kinetic term:

(I) \[ T_{\text{vib}}^{(I)} = \frac{1 + \kappa \beta^2}{2M} \left( \pi_x^2 + \pi_y^2 \right) \]

(II) \[ T_{\text{vib}}^{(II)} = \frac{1}{2M (1 + \kappa \beta^2)} \left( \pi_x^2 + \pi_y^2 \right) \]

Reasoning:
The Geometric Model in its basic form does not yield realistic moments of inertia:

\[ \Im_k(\beta, \gamma) = 4M\beta^2 \sin^2 \left( \gamma - \frac{2\pi}{3} k \right) \]

This can be improved by considering a deformation-dependent mass term: \( M = M(\beta) \).

In the Interacting Boson Model, similar (and much more complicated) kinetic terms arise naturally (see later in this Course).

In the following, we will only consider the influence of these generalized kinetic terms on \( J = 0 \) vibrations.
Simplified treatment of rotations

\[ H = \sum_{\beta, \gamma} \frac{J^2}{2 \Im T_{\text{rot}}} + \frac{1}{2M} \left( \frac{\pi^2}{\beta^2} + \frac{\pi^2_\beta}{\beta^2} \right) + ... + A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 + ... \]

\[ V = A(x^2 + y^2) + B(x^3 - 3y^2x) + C (x^2 + y^2)^2 > 0 \]

in the LAB frame

\[ J_\mu = i\sqrt{10}[\alpha \times \pi^*]_\mu^{(1)} \]

\[ J_1 = \frac{1}{\sqrt{2}} (-J_{+1} + J_{-1}) \]

\[ J_2 = \frac{1}{\sqrt{2}} (J_{+1} + J_{-1}) \]

\[ J_3 = J_0 \]

\[ T_{\text{rot}} = \frac{J^2_3}{8M\beta^2\sin^2\gamma} = \frac{M}{2} \beta^2 \sin^2\gamma \left( \frac{d\delta}{dt} \right)^2 \]

\[ J_3 = J'_3 \propto \frac{d\delta}{dt}_{\text{PAS}} \]

This treatment yields just a steady rotation. Even with this crucial simplification, we can study the basic influence of rotation on the vibration. It is substantial even for negligible angular momenta:

Evolution of 2 initially coincident trajectories
with \( j = 0 \) & \( j = \text{"very small"} \) in the deformation plane

Poincaré maps not applicable!
Quantization

\( J = 0 \)

\[
H = \frac{1}{2M} \left( \pi_x^2 + \pi_y^2 \right) + V
\]

\( V = A(x^2 + y^2) + B(x^3 - 3y^2x) + C(x^2 + y^2)^2 \)

\[
= A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4
\]

Two quantization options

2D system

\[
\hat{T}_{\text{vib}} = -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

\[
= -\frac{\hbar^2}{2M} \left( \frac{1}{\beta} \frac{\partial}{\partial \beta} \beta \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{\partial^2}{\partial \gamma^2} \right)
\]

with additional constraints

\[
\Psi(\beta, \gamma) = \Psi(\beta, \gamma + k\frac{2\pi}{3})
\]

(a) 2D even

\[
\Psi(\beta, \gamma) = \pm \Psi(\beta, -\gamma)
\]

(b) 2D odd

to avoid level crossings due to the symmetry of the potential

(c) 5D system restricted to 2D (true geometric model of nuclei)

\[
\hat{T}_{\text{vib}} = -\frac{\hbar^2}{2M} \left( \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right)
\]

Introduction to the 2 options differ also in the metric (measure) for calculating matrix elements

\[
\langle \Psi_1^{2D} | \Psi_2^{2D} \rangle = \int_0^{2\pi} \int_0 \Psi_1^{2D*} \Psi_2^{2D} \beta d\beta d\gamma
\]

\[
\langle \Psi_1^{5D} | \Psi_2^{5D} \rangle = \int_0^{2\pi} \int_0^{\pi} \Psi_1^{5D*} \Psi_2^{5D} \beta^4 |\sin 3\gamma| d\beta d\gamma
\]

Possibility to test quantum-classical correspondence in different quantization schemes
Quantization

$J = 0$

Numerical diagonalization in truncated 2D/5D oscillator bases. Convergence tested (=> fraction of $\sim \frac{1}{4}$ lower eigensolutions determined with sufficient accuracy).

$A = -1$

$B = 1.09$

$C = 1$

$\hbar^2 / M = 2.5 \cdot 10^{-3}$
Quantization

\( J = 0 \)

\[ A = -1, B = C = 1 \]
\[ \hbar^2 / M = 2.5 \times 10^{-3} \]
A paradise of chaos lovers!

$A = -1$
$C = 1$
$E = 0$
$J = 0$

$\beta = 0$ maximum of the potential $E = 0$

(schematic view)
A paradise of chaos lovers!

\( A = -0.84 \)

\( B, C, M = 1 \)

\( J = 0 \)
Map of classical chaos

Regular phase space fraction

\[ f_{\text{reg}} = \frac{\Omega_{\text{reg}}}{\Omega_{\text{tot}}} \]

- \( \beta_0 > 0 \)
- \( \beta_0 = 0 \)

Map of classical chaos


≈50,000 passages of 52 randomly chosen trajectories through the section $y=0$
Map of classical chaos

Increase of regularity at high energy due to the dominance of term $V \sim \beta^4$
Convex-concave transition

The diagram illustrates the change of the shape of the border of the accessible domain in the xy plane as the parameter \( \beta_0 \) changes. For \( \beta_0 > 0 \), the region is convex, and for \( \beta_0 = 0 \), it transitions to concave. The figure also shows the regular phase space fraction \( f_{\text{reg}} = \frac{\Omega_{\text{reg}}}{\Omega_{\text{tot}}} \), where \( \Omega_{\text{reg}} \) and \( \Omega_{\text{tot}} \) represent the areas of the regular and total domains, respectively. The transition from chaos to order is indicated by the change in the color map, with red representing chaos and blue representing order.
Test of Bohigas conjecture

for different quantization schemes & different Planck constants

\[ f_{\text{reg}} \ldots \text{classical regular fraction} \]
\[ 1-\omega \ldots \text{adjunct of Brody parameter} \]

\[ A = -1 \quad C = +1 \quad B = 0.24 \]

\[ \kappa = h^2 / M \]
\[ \kappa = 4 \cdot 10^{-6} \]
\[ \kappa = 9 \cdot 10^{-6} \]
\[ \kappa = 25 \cdot 10^{-6} \]

Stránský, Hruška, Cejnar
PRE 79, 046202, 066201 (2009)
Peres lattices

Correspondence to chaos

\[ \langle \psi_i | \hat{P} | \psi_i \rangle \equiv \langle P \rangle_i \]

1) Quasi-2D angular momentum

\[ L_{2D}^2 = \hbar^2 \frac{\partial^2}{\partial \gamma^2} \]

\[ L_{5D}^2 = \frac{\hbar^2}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \]

2) Hamiltonian perturbation

\[ H' = \beta^3 \cos 3\gamma \]

\[ V = -\beta^2 + B\beta^3 \cos 3\gamma + \beta^4 \]

\[ B = 1.09 \quad \kappa = \hbar^2 / M \]

Correspondence to chaos

1) \( < L^2 > \)

\[ \kappa = 4 \cdot 10^{-4} \]

2) \( < H' > \)

\[ \kappa = 25 \cdot 10^{-4} \]

\[ \kappa = 100 \cdot 10^{-4} \]
Peres lattices

The birth of chaos

\[ \langle \psi_i | \hat{P} | \psi_i \rangle \equiv \langle P \rangle_i \]

1) Quasi-2D angular momentum

\[ L_{2D}^2 = \hbar^2 \frac{\partial^2}{\partial \gamma^2} \]

\[ L_{5D}^2 = \frac{\hbar^2}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \]

2) Hamiltonian perturbation

\[ H' = \beta^3 \cos 3\gamma \]

\[ V = -\beta^2 + B \beta^3 \cos 3\gamma + \beta^4 \]

\[ \kappa = 25 \cdot 10^{-6} \]
Peres lattices

The birth of chaos

\[ \langle L^2 \rangle \]

1) \( B = 0 \)

2) \( < H' > \)

\[ B = 0 \]

\[ B = 0.001 \]

\[ B = 0.005 \]

\[ B = 0.05 \]

Stránský, Hruška, Cejnar
PRE 79, 046202, 066201 (2009)
Peres lattices

Regular / chaotic states

To some extent, the Peres method enables one to assign order/chaos to *individual states*
Peres lattices

Regular / chaotic states

To some extent, the Peres method enables one to assign order/chaos to individual states.
At low energies, near the main minimum of the potential, one can use the **harmonic-oscillator approximation**:

$$V \approx V_0 + \frac{M}{2} \omega_\beta^2 (\beta - \beta_0)^2 + \frac{M}{2} \omega_\gamma^2 \beta_0^2 (\gamma - \gamma_0)^2$$

The $\beta$ & $\gamma$ vibrations are separated and their interplay is the basic organization principle of low-energy spectra for $B > 0$ (away from $\gamma$ soft):

*Surprisingly*, this principle **remains valid at rather high energies!** Moreover, the main increase of regularity at $E \approx 0$ is located near the place with **equal frequencies**

$$\omega_\beta = \omega_\gamma$$

![Graphs and figures showing the energy spectra for different values of $B$.]( Diagrams showing the energy spectra for different values of $B$. )

**Numerology**

$$\tau = \frac{B^2}{AC}$$

- $\tau_1 = 2\tau_1$
- $\tau_2 = 3\tau_1$
- $\tau_3 = 4\tau_1$
Generalized cases

a) steady rotation

\[ J_3 = J'_3 \]

\[ \pi_x^2 + \pi_y^2 \]

b) extended Hamiltonian

\[ T^{(I)}_{vib} = \frac{1 + \kappa \beta^2}{2M} \left( \pi_x^2 + \pi_y^2 \right) \]

\[ T^{(II)}_{vib} = \frac{1}{2M(1 + \kappa \beta^2)} \left( \pi_x^2 + \pi_y^2 \right) \]
Physics of the 1st Kind
„Encoding“
Complex Behavior → Simple Equations →
Complex Behavior

Physics of the 2nd Kind
„Decoding“

INTERMEZZO

\[
\begin{align*}
\text{rot} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \text{div} \mathbf{D} &= \rho \\
\text{rot} \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} & \text{div} \mathbf{B} &= 0
\end{align*}
\]
Physics of the 1\textsuperscript{st} Kind

„Encoding“
Complex Behavior

→ Simple Equations →

X

Physics of the 2\textsuperscript{nd} Kind

„Decoding“
Complex Behavior

INTERMEZZO

Mandelbrot set
values of $c$ for which the complex sequence

$z_{n+1} = (z_n)^2 + c$

is bounded
Understanding Chaos via nuclei

Outline of the Course
1) Chaos in Classical Physics
2) Chaos in Quantum Physics
3) Chaos in Nuclei
4) Chaos in the Geometric Model
5) Chaos in the Interacting Boson Model

- a) Model
  - dynamical symmetries
  - geometric interpretation

- b) Map of chaos
  - “arc of regularity”

- c) A dynamical consequence of regularity
  - high-energy rotational bands
  - quasi dynamical symmetries
References

• **Dynamical-symmetry content of transitional IBM-1 hamiltonians**

• **Wave-function entropy and dynamical symmetry breaking in the interacting boson model**

• **Experimental confirmation of Alhassid-Whelan arc of regularity**

• **Classical and quantum properties of the semiregular arc inside the Casten triangle**

• **Order and chaos in the interacting boson model**

• **Peres lattices in nuclear structure**

• **Transition from gamma-rigid to gamma-soft dynamics in the interacting boson model: Quasicriticality and quasidynamical symmetry**

• **Occurrence of high-lying rotational bands in the interacting boson model**

• **Regularity-induced separation of intrinsic and collective dynamics**

• **Symmetry vs. chaos in collective dynamics**

• **Paar, Vorkapic, Dieperink**: PLB 205, 7 (1988); PRC 41, 2397 (1990), PRL 69, 2184 (1992)

• **Alhassid, Whelan, Novoselsky**: PRL 65, 2971 (1990); PRC 43, 2637 (1991); PRC 45, 1677 (1992); PRL 67, 816 (1991); NPA 556, 42 (1993)

• **Mizusaki, Yoshinaga, Shigehara, Cheon**: PLB 269, 6 (1991)

• **Canetta, Maino**: PLB 483, 55 (2000)

• **Macek, Leviatan**: PRC 84, 0413302 (2011), PLB 714, 110 (2012)
Interacting Boson Model

\[ \hat{H} = \sum_{i,j} u_{ij} b_i^+ b_j + \sum_{i,j,k,l} v_{ijkl} b_i^+ b_j^+ b_k b_l \]

\[ b_i^+ = \begin{cases} s^+ \\ d_{\mu}^+ & \mu = -2, \ldots, +2 \end{cases} \]

s-bosons \((l=0)\) \hspace{1cm} d-bosons \((l=2)\)

Interpretation:

- “nucleon pairs with \(l = 0, 2\)”
- “quanta of collective excitations”

Dynamical algebra: \( U(6) \)

Subalgebras: \( U(5), SO(6), SO(5), SO(3), SU(3) \)

The most general IBM Hamiltonian can be written as a weighted sum of Casimir invariants:

\[ \hat{H} = k_0 + k_1 \hat{C}_1[U(5)] + k_2 \hat{C}_2[U(5)] + k_3 \hat{C}_2[SO(6)] + k_4 \hat{C}_2[SO(5)] + k_5 \hat{C}_2[SO(3)] + k_6 \hat{C}_2[SU(3)] \]

Analytically solvable Hamiltonians:

- \( U(5) \): \( k_3 = k_6 = 0 \)
- \( SO(6) \): \( k_1 = k_2 = k_6 = 0 \)
- \( SU(3) \): \( k_1 = k_2 = k_3 = k_4 = 0 \)

**Interacting Boson Model**

**Dynamical Symmetry**

Complete chain of embeddings:

\[
G^{(D)} \supset G^{(1)} \supset G^{(2)} \cdots \supset G^{(S)}
\]

\[
\hat{H} = \hat{H}(\hat{C}^{(1)}, \hat{C}^{(2)}, \ldots, \hat{C}^{(S)})
\]

- The system’s Hilbert space \( \mathcal{H} \) = an eigenspace of Casimir invariant \( \hat{C}^{(D)} \) of the dynamical algebra
- Completeness: Casimir invariants \( \hat{C}^{(k)} \) of all subalgebras are compatible and the specification of all their eigenvalues determines a unique vector in \( \mathcal{H} \) (sometimes, like in IBM, there exist some missing labels)
- As a consequence, a **system with dynamical symmetry is integrable**!
- The opposite implication is not valid! Example: IBM Hamiltonian with no admixture of the SU(3) invariant is integrable but has no dynamical symmetry (**SO(5) “partial dynamical symmetry”**)

\[
\hat{H} = k_0 + k_1 \hat{C}_1^{[U(5)]} + k_2 \hat{C}_2^{[U(5)]} + k_3 \hat{C}_2^{[SO(6)]} + k_4 \hat{C}_2^{[SO(5)]} + k_5 \hat{C}_2^{[SO(3)]} + k_6 \hat{C}_2^{[SU(3)]}
\]

**Analytically solvable Hamiltonians:**

\[
U(6) \supset \left\{ \begin{array}{c} U(5) \\ SO(6) \\ SU(3) \end{array} \right\} \supset SO(5) \supset SO(3)
\]

- \( k_3 = k_6 = 0 \) for **U(5)**
- \( k_1 = k_2 = k_6 = 0 \) for **SO(6)**
- \( k_1 = k_2 = k_3 = k_4 = 0 \) for **SU(3)**

Simplified Hamiltonian

$$\hat{H}_{\eta, \chi} = a\left[ \eta \hat{n}_d - \frac{1-\eta}{N} \hat{Q}_\chi \cdot \hat{Q}_\chi \right]$$

scaling constant $a=1$

control parameters $\eta, \chi$ ensures “good behavior” for $N \to \infty$

$d$-boson number operator

$$\hat{n}_d = d^\dagger \cdot \tilde{d}$$

$$\hat{\mathcal{Q}}_\chi = [d^\dagger \times s + s^\dagger \times \tilde{d}]^{(2)} + \chi [d^\dagger \times \tilde{d}]^{(2)}$$

quadrupole operator

$$\hat{\mathcal{C}}_1[U(5)] = \hat{n}_d$$

$$\hat{\mathcal{C}}_2[SO(6)] = \hat{N}(\hat{N}+4) - \hat{P}_\pi^\dagger \hat{P}_\pi$$

$$\hat{\mathcal{C}}_2[SO(5)] = \hat{n}_d(\hat{n}_d+3) - (d^\dagger \cdot d^\dagger)(\tilde{d} \cdot \tilde{d})$$

$$\hat{\mathcal{C}}_2[U(5)] = \hat{n}_d(\hat{n}_d+4)$$

$$\hat{\mathcal{C}}_2[SO(3)] = 2 \hat{\mathcal{Q}} - \sqrt{7/2} \cdot \hat{\mathcal{Q}} - \sqrt{7/2} + \frac{3}{4} \hat{L} \cdot \hat{L}$$

$$\hat{\mathcal{C}}_2[SO(3)] = \hat{L} \cdot \hat{L}$$

**Geometric content**

Simplified Hamiltonian

\[ \hat{H}_{\eta,\chi} = a \left[ \eta \hat{n}_d - \frac{1 - \eta}{N} \hat{Q}_\chi \cdot \hat{Q}_\chi \right] \]

scaling constant \( a = 1 \)

control parameters \( \eta, \chi \) ensures “good behavior” for \( N \to \infty \)

**Correspondence to Geometric Model**

e.g.: Hatch, Levit, PRC 25, 614 (1982)

Use of **coherent states**

\[ |\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \exp \left( \alpha_s s^+ + \sum_{\mu} \alpha_\mu d^+ \right) |0\rangle \]

\[ \langle N \rangle \equiv \langle \alpha | \hat{N} | \alpha \rangle = |\alpha_s|^2 + \sum_{\mu} |\alpha_\mu|^2 \]

\( \Rightarrow \) shape variables \( \beta, \gamma \) & associated momenta \( p_\beta, p_\gamma \)

\( \Rightarrow \) classical **coordinate-momentum Hamiltonian**

\[ H_{\eta,\chi \text{cl}} = \frac{1}{2} \left[ \eta + 2(1-\eta)\beta^2 \right] \left( \beta^2 + T \right) - 2(1-\eta)\beta^2 \]

\[ - \frac{\chi(1-\eta)}{\sqrt{7}/2} \sqrt{1-(\beta^2+T)/2} \left[ \frac{p_\gamma}{\beta} - \beta p_\beta - \beta^3 \right] \cos 3\gamma + 2 p_\beta p_\gamma \sin 3\gamma \]

\[ - \frac{\chi^2(1-\eta)}{7/4} \left[ (\beta^2 + T)^2 / 8 - p_\gamma^2 / 2 \right] \]

\[ T = p_\beta^2 + \frac{p_\gamma^2}{\beta^2} \]

\( \beta \in [0, \sqrt{2}] \quad \gamma \in [0, 2\pi] \quad p_\beta \in [0, \sqrt{2}] \quad p_\gamma \in [0, 1] \)

... much more complicated kinetic terms than the Geometric Model
Geometric content

Simplified Hamiltonian

\[ \hat{H}_{\eta, \chi} = a \left[ \eta \hat{n}_d - \frac{1 - \eta}{N} \hat{Q}_\chi \cdot \hat{Q}_\chi \right] \]

scaling constant \( a = 1 \)

control parameters \( \eta, \chi \) ensures “good behavior” for \( N \to \infty \)

Correspondence to Geometric Model

e.g.: Hatch, Levit, PRC 25, 614 (1982)

Use of coherent states

\[ |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \exp(\alpha_s s^+ + \sum_\mu \alpha_\mu d^+) |0\rangle \]

\( \Rightarrow \) potential energy function in terms of shape variables

\[ V = A\beta^2 + B \sqrt{1 - \frac{\beta^2}{2}} \beta^3 \cos 3\gamma + \beta^4 \]

... similar as in the Geometric Model

\( \beta_0 = 0 \) spherical (I)
\( \beta_0 > 0 \) prolate (II)
\( \beta_0 < 0 \) oblate (III)

\( \varepsilon_0 = \min V \)
Map of chaos

More than 1 essential control parameter ⇒ multi-dimensional “map of chaos”

But there exist regions of almost full compatibility with the Geometric Model

\[ J = 0, \ E = 0 \]
Map of chaos

\[ 5.0 = \eta \]

\[ 7.0 = \eta \]

\[ \chi \]

\[ \text{totreg} / \Omega \]

\[ f \]

\[ \text{reg} \]

\[ \text{SU(3)} \]

\[ \text{U(5)} \]

\[ \text{SO(6)} \]

\[ \eta = 0.70 \]

\[ \eta = 0.50 \]

\[ \chi - \sqrt{7}/2 \]

\[ \chi 0 \]

\[ J = 0 \]

\[ f_{\text{reg}} = \Omega_{\text{reg}} / \Omega_{\text{tot}} \]

Macek, Cejnar, Jolie... 2007
Macek, Cejnar, Dobeš 2009
Macek (PhD) 2010

Map of chaos

\[ \eta = 0.5 \]

\[ \chi \]

\[ J = 0 \]

\[ f_{\text{reg}} = \Omega_{\text{reg}} / \Omega_{\text{tot}} \]
Map of chaos

“Arc of Regularity”

Discovered in:
See also: Alhassid, Whelan, Novoselsky
PRL 65, 2971 (1990); PRC 43, 2637 (1991);
PRC 45, 1677 (1992); NPA 556, 42 (1993)
Map of chaos

\[ \eta = \frac{1}{2}, \chi = 0 \]

\[ \eta = \frac{1}{2}, \chi = -0.23 \]

\[ \eta = \frac{1}{2}, \chi = -0.46 \]

\[ \eta = \frac{1}{2}, \chi = -0.68 \]

\[ \eta = \frac{1}{2}, \chi = -0.91 \]

\[ \eta = \frac{1}{2}, \chi = -1.16 \]

\[ f_{\text{reg}} = \frac{\Omega_{\text{reg}}}{\Omega_{\text{tot}}} \]

Macek, Cejnar, Jolie... 2007
Macek, Cejnar, Dobeš 2009
Macek (PhD) 2010

SU(3) \rightarrow U(5) \rightarrow SO(6)

integrable regime

arc of regularity
Map of chaos

\[ \eta = 0.5 \]

\[ \chi \]

\[ \text{SU(3)} \]
\[ \text{U(5)} \]
\[ \text{SO(6)} \]

\[ \eta = 0.8 \]

\[ \text{deformed} \]

\[ \text{spherical} \]

\[ f_{\text{reg}} = \frac{\Omega_{\text{reg}}}{\Omega_{\text{tot}}} \]

Macek, Cejnar, Jolie... 2007
Macek, Cejnar, Dobeš 2009
Macek (PhD) 2010

Integrable regime
Arc of regularity

Brody parameter

\[ J = 0 \text{ spectrum} \]
“Resonances”

Low $E$

- $\beta$ unstable
- $\gamma$ stable

stability change of $\beta$ & $\gamma$ vibrations

High $E$

- $\beta$ stable
- $\gamma$ unstable

stability change of some $\beta$ & $\gamma$ like vibrations ??

$\omega_\beta = \omega_\gamma$

$\eta = 0.5$

$J = 0$

$\Omega_{\text{reg}} / \Omega_{\text{tot}}$

$\eta = 0.7$
"Resonances"

**Low $E$**
- $\beta$ unstable
- $\gamma$ stable
- Stability change of $\beta$ & $\gamma$ vibrations

**High $E$**
- $\beta$ stable
- $\gamma$ unstable
- Stability change of some $\beta$ & $\gamma$ like vibrations ???

$J = 0$

$\omega_\beta = \omega_\gamma$

$E = \frac{2}{11}(E_{\text{max}} - E_{\text{min}})$
Physics of Chaos not only keeps the physicists busy...

... but also makes their children happy!

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Quasi Dynamical Symmetry: Exact dynamical symmetry is broken in such a way that eigenstates of the Hamiltonian become coherent mixtures of irreducible representations of the original exact symmetry.

Example: **SU(3) QDS** close to the SU(3)→U(5) critical point in IBM (N=100)

\[ |l_i\rangle = \sum_{(\lambda,\mu),K} A_{(\lambda,\mu),K}^l \langle (\lambda,\mu), K, l | \]

Rotational bands decomposition of wavefunctions in the SU(3) basis

From: Rosensteel, Rowe (2005)
Rotational bands

\[ \hat{H}_{(\eta,\chi)} = (1-\eta) \left[ -\frac{\hat{Q}_x \cdot \hat{Q}_x}{N} \right] + \eta \hat{n}_d \]

\(N=30 \Rightarrow\)
- 91 states 0^+
- 165 states 2^+
- 225 states 4^+
- 271 states 6^+

actual \(i^{th}\) eigenstate with angular momentum \(l\)

\[ |l_i\rangle = \sum_{(\lambda,\mu),K} A_{(\lambda,\mu),K}^{l_i} |(\lambda,\mu),K,l\rangle \]

decomposition of wavefunctions in the SU(3) basis

SU(3) eigenstate

probability

\[ P_{(\lambda,\mu)}^{l_i} = \sum_K \left| A_{(\lambda,\mu),K}^{l_i} \right|^2 \]
**Rotational bands**

\[ \eta = 0.5, \chi = -0.5 \]

\[ \eta = 0.5, \chi = -0.9 \]

\[ \hat{H}_{(\eta,\chi)} = (1 - \eta) \left[ -\frac{\hat{Q}_x \cdot \hat{Q}_y}{N} \right] + \eta \hat{n}_d \]

\[ C(2,4)_i = \max_j [\pi(0_i, 2_j)] \]
\[ \times \max_k [\pi(0_i, 4_k)] \]

\[ N = 30 \]

**Correlation coefficient** of the SU(3) decompositions for 0\(_i\) and \(l_j\) states = \(\pi(0_i, l_j)\)

\[ \pi(\bar{x}, \bar{y}) = \frac{1}{n-1} \sum_k \frac{(x_k - \bar{x})(y_k - \bar{y})}{s_x s_y} \]

Candidates for rotational bands: the \(l_j\) states above 0\(_i\) having **maximal** \(\pi(0_i, l_j)\)

\[ |l_i\rangle = \sum_{(\lambda,\mu),K} A_{(\lambda,\mu),K}^{l_i} |(\lambda, \mu), K, l\rangle \]

Actual \(i\)th eigenstate with angular momentum \(l\)

decomposition of wavefunctions in the SU(3) basis

\[ P_{(\lambda,\mu)}^{l_i} = \sum_K |A_{(\lambda,\mu),K}^{l_i}|^2 \]

SU(3) eigenstate

**Probability**
\[ H_{(\eta, \chi)} = (1 - \eta) \left( -\frac{\hat{Q}_x \cdot \hat{Q}_x}{N} \right) + \eta \hat{n}_d \]

\[ C(2,4)_i = \max_j [\pi(0,2_j)] \times \max_k [\pi(0,4_k)] \]

\[ N = 30 \]

Moment of inertia for a given band:

\[ E(l) - E(0) = \frac{1}{2J_l} l(l+1) \]

For true bands: \( J_2 \approx J_4 \approx \ldots \)
Rotational bands

\[ \eta = 0.5, \chi = -0.5 \]

\[ \eta = 0.5, \chi = -0.9 \]

\[ \hat{H}_{(\eta, \chi)} = (1 - \eta) \left[ -\frac{\hat{Q}_x \cdot \hat{Q}_x}{N} \right] + \eta \hat{n}_d \]

\[ C(2,4)_i = \max_j [\pi(0_i,2_j)] \times \max_k [\pi(0_i,4_k)] \]

\[ N = 30 \]

\[ R(4/2)_i = \frac{E(4_k) - E(0_i)}{E(2_j) - E(0_i)} \]

Alaga rule

\[ B^{E2}(l, K \to l', K') = (lK2\Delta K | l' K')^2 \langle K' | T^{E2} | K \rangle \]

\[ \Rightarrow B^{E2}(l_1, K \to l'_1, K') = \frac{(l_1 K 2\Delta K | l'_1 K')^2}{(l_2 K 2\Delta K | l'_2 K')^2} \]

works very well!
Adiabatic separation of intrinsic & collective dynamics

$\eta = 0.5$, $\chi = -0.5$

$\eta = 0.5$, $\chi = -0.9$

$\hat{H}_{(\eta, \chi)} = (1 - \eta) \left[ -\frac{\hat{Q}_x \cdot \hat{Q}_x}{N} \right] + \eta \hat{n}_d$

$C(2,4)_i = \max_j [\pi(0_i, 2_j)] \times \max_k [\pi(0_i, 4_k)]$

$N = 30$

$R(4/2)_i = \frac{E(4_k) - E(0_i)}{E(2_j) - E(0_i)}$

$\Omega_{\text{reg}} / \Omega_{\text{tot}}$

Increased regularity of intrinsic dynamics $\Rightarrow$ Adiabatic separation of intrinsic & collective dynamics

More rotational bands

Indication of the SU(3) QDS
Conclusions

• Both Sphairos and Chaos follow some universal mathematical laws and their competition belongs to the most fundamental topics in physics
• Atomic nuclei constitute an exemplary realization of chaotic dynamics in the quantum domain
• Nuclear collective motions exhibit an intricate interplay of regular and chaotic features
  => a new paradigm of chaos (testing ground for general studies)
• Regularity may have unexpected dynamical consequences, like the emergence of adiabatic separation of intrinsic and collective motions

Acknowledgments

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Thank You for patience!
I had a feeling once about Mathematics, that I saw it all—Depth beyond depth was revealed to me—the Byss and the Abyss. I saw, as one might see the transit of Venus—or even the Lord Mayor's Show, a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how it happened and why the tergiversation was inevitable: and how the one step involved all the others. It was like politics. But it was after dinner and I let it go!

Winston Chirchill, *My Early life: 1874 - 1904*