The study of shape phase transitions in nuclei and related models may enrich the general theory of quantum phase transitions.

1. The influence of additional degrees of freedom
   We can study these phenomena in coupled versions of Our Models, such as proton-neutron IBM, configuration-mixed IBM, Bose-Fermi IBFM…

2. Mechanisms of first-order and continuous QPTs
   Our Models include both basic types of QPT within the same framework. We can perform comparative studies of scaling properties, complex extensions…

3. Quantum phase transitions for excited states
   Our Models exhibit non-analytic evolutions of excited states related to non-analytic changes of classical dynamics. We can analyze these effects to get deeper insight into the connection of QPTs with thermal phase transitions.
“Fundamental aspects”
of nuclear shape phase transitions

Pavel Cejnar
Inst. of Particle and Nuclear Physics, Charles Univ., Prague, CZ
cejnar @ ipnp.troja.mff.cuni.cz

The study of shape phase transitions in nuclei and related models may enrich the general theory of quantum phase transitions.

1. The influence of additional degrees of freedom
   We can study these phenomena in coupled versions of Our Models, such as proton-neutron IBM, configuration-mixed IBM, Bose-Fermi IBFM...

2. Mechanisms of first-order and continuous QPTs
   Our Models include both basic types of QPT within the same framework. We can perform comparative studies of scaling properties, complex extensions...

3. Quantum phase transitions for excited states
   Our Models exhibit non-analytic evolutions of excited states related to non-analytic changes of classical dynamics. We can analyze these effects to get deeper insight into the connection of QPTs with thermal phase transitions.

Istanbul, September 2009
1. Extra degrees of freedom

\[ H = (1 - \zeta) n_d - \frac{1}{N} \zeta (Q^\pi \cdot Q^\nu) \]
\[ Q_s = d^s + s^d + \chi [d^s + s^d]^{(2)} \]

proton-neutron IBM

\[ H = (1 - \zeta) (n^\pi_d + n^\nu_d) - \frac{1}{N} \zeta (Q^\pi_{\chi^\pi} + Q^\nu_{\chi^\nu}) \cdot (Q^\pi_{\chi^\pi} + Q^\nu_{\chi^\nu}) \]
\[ = H^\pi + H^\nu - \frac{2 \zeta}{N} (Q^\pi_{\chi^\pi} \cdot Q^\nu_{\chi^\nu}) \]

coherent-state method

\[ |\Psi\rangle \propto (\Gamma^+_{\pi\pi})^N_{\pi} |0_{\pi}\rangle \otimes (\Gamma^+_{\nu\nu})^N_{\nu} |0_{\nu}\rangle \]

minimization of \( V = \langle \Psi | H | \Psi \rangle \) in \( \beta^\pi_{\pi}, \gamma^\pi_{\pi}, \beta^\nu_{\nu}, \gamma^\nu_{\nu} \)

Caprio, Iachello (2004, 2005)

combined systems

\[ H = H_1 \otimes H_2 \]
\[ H = H_1 + H_2 + H_{12} \]
1. Extra degrees of freedom

IBFM with $j \geq 3/2$ fermion


$$H = H^B + H^F - \kappa \left( Q^B_{\chi_f} \cdot [a^+_j \times \tilde{a}^+_j]^{(2)} \right)$$

coherent-state method

$$|\Psi\rangle \propto (\Gamma^+_{\beta \gamma})^N |0_B\rangle \otimes \left( \sum_m c^+_m a^+_m \right) |0_F\rangle$$

minimization of $V = \langle \Psi | H | \Psi \rangle$ in $\beta, \gamma, c_m$

$$V = \frac{A \beta^2 + B \beta^3 \cos 3\gamma + C \beta^4}{(1 + \beta^2)^2} - \kappa' \sqrt{\frac{\beta^2 - \bar{\chi} \beta^3 \cos 3\gamma + \frac{\bar{\chi}^2}{4} \beta^4}{(1 + \beta^2)^2}}$$

combined systems

$$H = H_1 \otimes H_2$$

$$H = H_1 + H_2 + H_{12}$$

Example: SU(3) bosonic core $\times$ 1 fermion $j = 3/2$

Bosonic core quadrupole parameter

Boson-fermion interaction quadrupole parameter

If $\text{sig } \chi_B \neq \text{sig } \chi_F$:

polarization of the core to a triaxial shape and to the opposite axial shape

$$\bar{\chi} = \sqrt{\frac{2}{7}} \chi_F$$

$$\kappa' = \frac{1}{\sqrt{3}} | \kappa |$$

$\kappa' = 4$ triaxial

$\kappa' = 1$ prolate

$\kappa' = 7$ oblate
1. Extra degrees of freedom

\[ H = (1 - \zeta) n_d - \frac{1}{N} \zeta (Q_\chi \cdot Q_\chi) \]

\[ Q_\chi = d^+ s + s^+ \tilde{d} + \chi [d^+ \tilde{d}]^{(2)} \]

Combined systems

\[ H = H_1 \otimes H_2 \]

\[ H = H_1 + H_2 + H_{12} \]

Additional types of bosons (e.g., sdg-IBM)

Devi, Kota (1990)

Additional terms in the Hamiltonian

Higher-order (3-body) boson interactions

External rotation

Extended systems

\[ H \text{ unchanged} \]

\[ H = H_0 + V \]

Different configuration space (other degrees of freedom than \( \beta, \gamma \), higher deformations)

Triaxial phase
1. Extra degrees of freedom

**Configuration mixing in IBM**

$H_1 \ldots$ sd-space for $N$ bosons… projector $P_1$

$H_2 \ldots$ sd-space for $N+2$ bosons… projector $P_2$

coherent-state method

$$|\Psi\rangle = c_1 P_1 (\Gamma_{\beta\gamma}^+)^N |0\rangle + c_2 P_2 (\Gamma_{\beta\gamma}^+)^{N+2} |0\rangle$$

minimization of $V = \langle \Psi | H | \Psi \rangle$ in $\beta, \gamma, c_1, c_2$

$V = (c_1^* c_2) \left( \begin{array}{c} \langle \Psi_1 | H_1 | \Psi_1 \rangle \\ \langle \Psi_2 | H_{\text{mix}} | \Psi_1 \rangle \\ \langle \Psi_2 | H_2 | \Psi_2 \rangle \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right)$

This setup yields phase structures beyond IBM, e.g., a possibility of prolate-oblate shape coexistence.*

**combined systems**

$$H = H_1 \oplus H_2$$

$$H = P_1 H_1 P_1 + P_2 H_2 P_2 + H_{\text{mix}}$$

De Coster, Heyde et al. (1996-1999)

Frank, Van Isacker, Vargas (2004)
Frank, Van Isacker, Iachello (2006)
Morales et al. (2008)
Hellemans et al. (2007, 2009)
The study of shape phase transitions in nuclei and related models may enrich the general theory of quantum phase transitions.

1. The influence of additional degrees of freedom
   We can study these phenomena in coupled versions of Our Models, such as proton-neutron IBM, configuration-mixed IBM, Bose-Fermi IBFM…

2. Mechanisms of first-order and continuous QPTs
   Our Models include both basic types of QPT within the same framework. We can perform comparative studies of scaling properties, complex extensions…

3. Quantum phase transitions for excited states
   Our Models exhibit non-analytic evolutions of excited states related to non-analytic changes of classical dynamics. We can analyze these effects to get deeper insight into the connection of QPTs with thermal phase transitions.
2. QPT mechanisms

**Energy gap**

\[ \Delta = E_{1^{\text{st}} \text{excited}} - E_{\text{ground}} \]

\[ \tau = \frac{\hbar}{\Delta} \]
determines a characteristic time-scale of the system

---

**Example:**

\[ \Delta_c \propto e^{-aN} \]

\[ \Delta_c \propto N^{-1/3} \]

---

Lipkin Hamiltonian: \[ H = \eta n_t - \frac{1}{N} (1 - \eta) Q_{\chi} Q_{\chi} \]

\[ n_t = t^+ t, \quad Q_{\chi} = t^+ s + s^+ t + \chi \quad t^+ t \]

Vidal *et al.*, PRC 73, 054305 (2006)

---

Need for large-$N$ calculations  $\Rightarrow$ methods beyond the mean field

Results for two-level boson models: Dusuel, Vidal, Arias, Dukelsky, García-Ramos (2005-2007)
2. QPT mechanisms

**finite-size scaling properties**

**Example: Energy gap**

\[ \Delta = E_{\text{1st excited}} - E_{\text{ground}} \]

\[ \tau = \frac{\hbar}{\Delta} \]

2. **QPT mechanisms**

* finite-size scaling properties

**Example: Energy gap**

\[ \Delta = E_{\text{1st excited}} - E_{\text{ground}} \]

\[ \tau = \frac{\hbar}{\Delta} \]

1st order

\[ \Delta_c \propto e^{-aN} \]

There is no global scaling of the whole spectrum at the critical point, but the spectrum per boson consists of “parity” doublets with spacings \( \Delta_\gamma \approx \hbar = N^{-1} \)

\[ \Delta_\delta \approx \exp\left(-\frac{a}{\hbar}\right) = \exp\left(-aN\right) \]

Quantum tunneling

2nd order

At the critical point the whole spectrum per boson scales as \( N^{-4/3} \).

\[ \Delta_c \propto N^{-4/3} \]

Pure quartic oscillator scales as \( \hbar^{4/3} \)

Rowe et al. (2004)

1D case

\[ V \propto \frac{1}{11} x^2 - \frac{4\sqrt{2}}{11} x^3 + \frac{8}{11} x^4 \]

\[ \rho_{\text{low}}(\epsilon, N) \propto N \]

\[ \rho_{\text{low}}(\epsilon, N) \propto N^{\frac{4}{3}} \epsilon^{-\frac{1}{3}} \]

Rowe et al. (2004)
2. QPT mechanisms

\[ H = H_0 + \lambda \dot{H} \]

complex extension

\[ \lambda \rightarrow \Lambda = \lambda + i\mu \]

Degeneracy of Hamiltonian eigenvalues:

- real case \(\Rightarrow\) rear solutions
- complex case \(\Rightarrow\) abundant solutions

Complex degeneracies determine the dynamics of real energy levels!

- Kato (1966)
- Zirnbauer, Verbaaschot, Weidenmüller (1983)
- Shanley (1988)
- Cejnár, Heinze, Dobeš (2005), Cejnár, Heinze, Macek (2007)
2. QPT mechanisms

**complex extension**

\[ H = H_0 + \lambda \dot{H} \]

\[ \lambda \rightarrow \Lambda = \lambda + i \mu \]

**2 x 2 example:**

\[
H = \begin{pmatrix}
e_1 & v \\
v & e_2
\end{pmatrix} + \Lambda \begin{pmatrix}
\dot{e}_1 & \dot{v} \\
\dot{v} & \dot{e}_2
\end{pmatrix} = \begin{pmatrix}
e_1 + \Lambda \dot{e}_1 & v + \Lambda \dot{v} \\
v + \Lambda \dot{v} & e_2 + \Lambda \dot{e}_2
\end{pmatrix} = \begin{pmatrix}
E_1 & V \\
V & E_2
\end{pmatrix}
\]

\[ E_{\pm} = \frac{1}{2} \left[ E_1 + E_2 \pm \Delta E \right] \]

\[ \Delta E = \sqrt{(E_1 - E_2)^2 + 4V^2} = \sqrt{a + b \Lambda + c \Lambda^2} \]

**Degeneracy \( \Delta E = 0 \)**

- **real case**: 2 conditions \( \Rightarrow \) rear solutions in \( \lambda \)
- **complex case**: 1 condition \( \Rightarrow \) solution in \( \Lambda \) always exists

Local behavior close to:

- complex degeneracy \( \Lambda_0 = \lambda_0 \pm i \mu_0 \)
- real degeneracy \( \Lambda_0 = \lambda_0 \pm i0 \) (if any)

**Passing a branch point along the real axis:**

\[ \Delta E \propto \sqrt{(\lambda - \lambda_0)^2 + \mu_0^2} \]

Avoided crossing of levels

Actual crossing

Branch point \( \mu_0 = 0.05 \)

Branch point \( \mu_0 = 0.01 \)

Diabolic point \( \mu_0 = 0 \)
2. QPT mechanisms

\[ H = H_0 + \lambda \dot{H} \]

\[ \lambda \rightarrow \Lambda = \lambda + i \mu \]

Complex extension

**2 \times 2 example:**

\[ H = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} + \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Sharpness of the crossing indicated by:

\[ C_0 \propto \frac{d^2}{d\lambda^2} \ln R = \frac{d^2}{d\lambda^2} \ln |E_+ - E_-| \]

Peak area:

\[ \text{fwhm} \times \text{height} \propto v^{-1} \propto \Delta_c^{-1} \]

Cejnàr, Heinze, Macek (2007)

**Example:**

\[ v \rightarrow 0 \Rightarrow \text{"1st order QPT"} \]

\[ C_0 \propto \frac{v^2 - \lambda^2}{(v^2 + \lambda^2)^2} \]

"2D electrostatic potential" 

"Coulomb force gradient"
2. QPT mechanisms

**Cusp Hamiltonian:** 1\(^{\text{st}}\) order QPT

\[ H = H_0 + \lambda \dot{H} \]

Complex extension

\[ \lambda \rightarrow \Lambda = \lambda + i\mu \]

**Many-level system**

\[ C_0 = \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_i = \frac{d^2}{d\lambda^2} \sum_{i>0} \ln |E_i - E_0| \]

50 levels included in the calculation, only the closest one gives an essential contribution

⇒ **1\(^{\text{st}}\) order QPT is locally a 2-level process!**

\[ \text{Peak area: } \text{fwhm} \times \text{height} \propto \exp(a / \hbar) \propto \Delta^{-1} \]

Cejnar, Heinze, Macek (2007)

All branch points located on the \(E(\Lambda)\) **Riemann sheet** corresponding to the ground state solution \(E_0(\Lambda)\).
2. QPT mechanisms

\[ H = H_0 + \lambda \dot{H} \]

\[ \lambda \rightarrow \Lambda = \lambda + i\mu \]

**Cusp Hamiltonian: 2\textsuperscript{nd} order QPT**

\[ H = -\frac{\hbar^2}{2} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

Size parameter = $\hbar^{-1}$

50 levels included in the calculation, giving only slowly decreasing contributions

⇒ **Cont. QPT is locally a many-level process!**

- **Cusp Hamiltonian**:
  - \[ C_0 = \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_i = \frac{d^2}{d\lambda^2} \sum_{i>0} \ln|E_i - E_0| \]

- **Peak area**:
  - \[ \text{fwhm} \times \text{height} \propto \hbar^{-2/3} \propto \Delta_c^{-1/2} \]

- All branch points located on the $E(\Lambda)$ **Riemann sheet** corresponding to the ground state solution $E_0(\Lambda)$

Cejnar, Heinze, Macek (2007)
2. QPT mechanisms

Cusp Hamiltonian: 2\textsuperscript{nd} order QPT

\[ H = H_0 + \lambda \dot{H} \]

size parameter = \( \hbar^{-1} \)

50 levels included in the calculation, giving only slowly decreasing contributions

\[ \Rightarrow \text{Cont. QPT is locally a many-level process!} \]

Cusp Hamiltonian: 2\textsuperscript{nd} order QPT

\[ \lambda \rightarrow \Lambda = \lambda + i\mu \]

The number of strongly contributing terms in the sum is proportional to \( \hbar^{-1} \)

\[ C_0 = \hbar \frac{d^2}{d\lambda^2} \sum_{i>0} \ln R_i = \hbar \frac{d^2}{d\lambda^2} \sum_{i>0} \ln |E_i - E_0| \]

\[ \text{Peak area: } \text{fwhm} \times \text{height} \propto \hbar^{1/3} \propto \Delta_c \]

The scaling factor does not influence the exponential increase for the 1\textsuperscript{st} order QPT \( \Rightarrow \) infinite “latent heat”
2. QPT mechanisms

**IBM O(6)-U(5) Hamiltonian:** 2\textsuperscript{nd} order QPT

\[ H = H_0 + \lambda \dot{H} \]

\[ \lambda \rightarrow \Lambda = \lambda + i\mu \]

**scaling factor**

**complex extension**

\[ \frac{H}{N} = \eta \frac{n_d - 1 - \eta}{N^2} Q_0 \cdot Q_0 \]

size parameter \( \hbar^{-1} \Leftrightarrow N \)

\( n_d = d^+ \tilde{d} \quad Q_\chi = d^+ s + s^+ \tilde{d} + \chi [d^+ \times \tilde{d}]^{(2)} \)

\[ C_0 = \frac{1}{n-1} \frac{d^2}{d\lambda^2} \sum_{i=1}^{n} \ln R_i = \frac{1}{n-1} \frac{d^2}{d\lambda^2} \sum_{i=1}^{n} \ln |E_i - E_0| \]

\( n \propto N \)

**Thermodynamic analogy:**

- Scaled \( C_0 \) is an analog of "specific heat"
- Branch points are like complex zeros of partition function [Yang, Lee 1952]

Cejnar, Heinze, Doeš 2005

- Zero "latent heat" in 2\textsuperscript{nd} order phase transition

**Graph:**

- Peak area:
- fwhm x height
- 0\textsuperscript{+} states with seniority=0

\[ C_0 \propto N \]

\[ \eta \propto N \]

\[ n = \text{no. of levels involved (dimension)} \]

\[ n-1 = \text{no. of branch points on the g.s. Riemann sheet} \]

\[ \hbar \rightarrow 0 \]

\[ \text{fwhm x height } \propto N^{-1/3} \propto \Delta_c \]

Cejnar, Heinze, Macek (2007)
The study of shape phase transitions in nuclei and related models may enrich the general theory of quantum phase transitions.

1. The influence of additional degrees of freedom
   We can study these phenomena in coupled versions of Our Models, such as proton-neutron IBM, configuration-mixed IBM, Bose-Fermi IBFM...

2. Mechanisms of first-order and continuous QPTs
   Our Models include both basic types of QPT within the same framework. We can perform comparative studies of scaling properties, complex extensions...

3. Quantum phase transitions for excited states
   Our Models exhibit non-analytic evolutions of excited states related to non-analytic changes of classical dynamics. We can analyze these effects to get deeper insight into the connection of QPTs with thermal phase transitions.
3. **Excited-state quantum phase transitions**

thermal phase transition ⇔ a singular evolution of the spectrum with the control parameter

\[ \hat{H} = \hat{H}_0 + \lambda \hat{H}' \]

Free energy

\[ F = \text{Tr}(\hat{\mathcal{Q}} \hat{H}) + T \text{Tr}(\hat{\mathcal{Q}} \ln \hat{\mathcal{Q}}) = -T \ln Z \]

\[ \langle E \rangle \quad \text{thermal average of energy} \]

\[ -S \quad \text{entropy} \]

\[ \text{partition function} \]

\[ \frac{\partial F}{\partial T} = -S, \quad \frac{\partial^2 F}{\partial T^2} = -\frac{\langle \langle E^2 \rangle \rangle}{T^3} \]

\[ \langle \langle XY \rangle \rangle = \langle XY \rangle - \langle X \rangle \langle Y \rangle \]

\[ \dot{E}_i \equiv \frac{d}{d\lambda} E_i, \quad \ddot{E}_i \equiv \frac{d^2}{d\lambda^2} E_i \]

\[ S \propto \ln \rho \]

\[ \langle \dot{E} \rangle, \quad \frac{\partial^2 F}{\partial T \partial \lambda} = \frac{\langle E \dot{E} \rangle}{T^2} \]

Cejnàr, Stránský (2008)
3. Excited-state quantum phase transitions

thermal phase transition ⇔ a singular evolution of the spectrum with the control parameter

\[ \hat{H} = \hat{H}_0 + \lambda \hat{H}' \]

Finite quantum systems (systems with a finite number of quantum degrees of freedom):

\[ N \to \infty \iff \hbar \to 0 \]

Singular growth of the level density

- 1\textsuperscript{st} order: discontinuity
- 2\textsuperscript{nd} order: discontinuous 1\textsuperscript{st} derivative
- ... 
- \(n\)\textsuperscript{th} order: discontinuous \((n-1)\textsuperscript{th}\) derivative
- continuous no order: singular derivative

Singular evolution of the spectrum (anomalous level dynamics)

sharp “collisions” (avoided crossings) of many levels

individual levels may show coherent or chaotic dynamics

Cejnar, Stránský (2008)
3. Excited-state quantum phase transitions

\[ H = \eta n_d - \frac{1}{N} (1 - \eta) \left( Q_{\chi=0} \cdot Q_{\chi=0} \right) \]

Heinze, Macek, Cejnar, Jolie, Dobeš (2006)
Cejnar, Heinze, Macek (2007)

\[ C_k = \frac{1}{n-1} \frac{d^2}{d\lambda^2} \sum_{i \neq k} \ln |\mathcal{E}_i - \mathcal{E}_k| \]

fwhm × height \( \propto N^{-m} \)

\( fwhm \times height \propto N^{-m} \)

\( x = 0.1 \)

\[ \rho \]

1D

2D

\( L = 0 \)

\( v = 0 \)

\( v \geq 0 \)

QPT (2\textsuperscript{nd} order)

ESQPT (continuous) precursors scale with \( \nu/N \)

\( N=100 \)

\( L=0 \)

seniority \( \nu = 0 \)
3. Excited-state quantum phase transitions

\[ H = -\frac{1}{N} \left( Q_{\chi=-\sqrt{\frac{7}{2}}} \cdot Q_{\chi=-\sqrt{\frac{7}{2}}} \right) \]

\[ Q_{\chi} = d^+ s + s^+ d + \chi [d^+ d]^2 \]

\[ \mathcal{E} = -\frac{1}{2} \left[ \left( \frac{\lambda}{N} \right)^2 + \left( \frac{\mu}{N} \right)^2 + \frac{\lambda}{N} \frac{\mu}{N} + \frac{3}{N} \frac{\lambda}{N} + \frac{3}{N} \frac{\mu}{N} \right] \]

\[(\lambda, \mu) = (2N,0), (2N-4,2), (2N-8,4), \ldots \]

\[(2N-6,0), (2N-10,2), (2N-14,4), \ldots \]

\[(2N-12,0), (2N-16,2), (2N-18,4), \ldots \]

\[N=20, \quad L=0 \text{ states} \]

\[\mathcal{E} = -0.5 \quad \mathcal{E} = -1 \quad \mathcal{E} = -2 \]

\[N \rightarrow \infty \]

Number of states with energy \(\leq \mathcal{E}\) is prop. to area \(S\)

\[\rho \propto \frac{dS}{d\mathcal{E}}\]
3. Excited-state quantum phase transitions

\[ H = -\frac{1}{N} \left( Q_{\chi=-\frac{\sqrt{7}}{2}} \cdot Q_{\chi=-\frac{\sqrt{7}}{2}} \right) \]

\[ Q_{\chi} = d^+ s + s^+ \tilde{d} + \chi [d^+ \tilde{d}]^2 \]

\[ \mathcal{E} = -\frac{1}{2} \left[ \left( \frac{\lambda}{N} \right)^2 + \left( \frac{\mu}{N} \right)^2 + \frac{\lambda}{N} \frac{\mu}{N} + \frac{3}{N} \frac{\lambda}{N} + \frac{3}{N} \frac{\mu}{N} \right] \]

\[ (\lambda, \mu) = (2N, 0), (2N - 4, 2), (2N - 8, 4), \ldots \]

\[ (2N - 6, 0), (2N - 10, 2), (2N - 14, 4), \ldots \]

\[ (2N - 12, 0), (2N - 16, 2), (2N - 18, 4), \ldots \]

\[ N=20 \]

\[ L=0 \] states

Phase transition at \( \mathcal{E}=-0.5 \)

\[ \rho \text{ continuous} \Rightarrow \text{ softer than 1}^{\text{st}}\text{order} \]

\[ \frac{d\rho}{d\mathcal{E}} \text{ singular} \Rightarrow \text{ harder than 2}^{\text{nd}}\text{order} \]

\[ \Rightarrow \text{ continuous phase transition with no Ehrenfest order} \]

\[ N \rightarrow \infty \]

\[ \rho \propto \frac{dS}{d\mathcal{E}} \]

level density
3. Excited-state quantum phase transitions

\[ H = -\frac{1}{N} \left( Q_{\chi = -\sqrt{\frac{1}{2}}} \cdot Q_{\chi = -\sqrt{\frac{1}{2}}} \right) \]

\[ \mathcal{E} = -\frac{1}{2} \left[ \left( \frac{\lambda}{N} \right)^2 + \left( \frac{\mu}{N} \right)^2 + \frac{\lambda}{N} \cdot \frac{\mu}{N} + \frac{3}{N} \cdot \frac{\lambda}{N} + \frac{3}{N} \cdot \frac{\mu}{N} \right] \]

\[ Q_{\chi} = d^+ s + s^+ \tilde{d} + \chi [d^+ \tilde{d}]^{(2)} \]

\[ V = -\frac{4\beta^2 + 2\sqrt{2}\beta^3 \cos 3\gamma + \frac{1}{2} \beta^4}{(1 + \beta^2)^2} \]

Phase transition at \( \mathcal{E} = -0.5 \)
- \( \rho \) continuous \( \Rightarrow \) softer than 1\textsuperscript{st} order
- \( \frac{d\rho}{d\mathcal{E}} \) singular \( \Rightarrow \) harder than 2\textsuperscript{nd} order
- \( \Rightarrow \) continuous phase transition with no Ehrenfest order

\[ \rho(E) \propto \int \delta(E - H) \, d^f x \, d^f p = \frac{d}{dE} \int \Theta(E - H) \, d^f x \, d^f p \]

level density \( \propto \) phase-space volume

SU(3) limit of IBM1

Phase transition at \( \mathcal{E} = -0.5 \)
- \( \rho \) continuous \( \Rightarrow \) softer than 1\textsuperscript{st} order
- \( \frac{d\rho}{d\mathcal{E}} \) singular \( \Rightarrow \) harder than 2\textsuperscript{nd} order
- \( \Rightarrow \) continuous phase transition with no Ehrenfest order

\[ \rho \]

\[ N \to \infty \]

singular growth
3. Excited-state quantum phase transitions

\[ H = -\frac{1}{\mathcal{N}} \left( \mathcal{Q}_{\chi = -\frac{\sqrt{7}}{2}} \cdot \mathcal{Q}_{\chi = -\frac{\sqrt{7}}{2}} \right) \]

\[ \mathcal{E} = -\frac{1}{2} \left[ \left( \frac{\mathcal{A}}{\mathcal{N}} \right)^2 + \left( \frac{\mathcal{H}}{\mathcal{N}} \right)^2 + \frac{\mathcal{A}}{\mathcal{N}} \frac{\mathcal{H}}{\mathcal{N}} + \frac{3}{\mathcal{N}} \frac{\mathcal{A}}{\mathcal{N}} + \frac{3}{\mathcal{N}} \frac{\mathcal{H}}{\mathcal{N}} \right] \]

**SU(3) limit of IBM1**

Phase transition at \( \varepsilon = -0.5 \)

- \( \rho \) continuous \( \Rightarrow \) softer than 1\(^{\text{st}}\) order
- \( \frac{d\rho}{d\varepsilon} \) singular \( \Rightarrow \) harder than 2\(^{\text{nd}}\) order

**Finite-\( N \) realization:**

- \( N = 60 \)

**\( N \to \infty \)**

- \( \rho \) continuous phase transition with no Ehrenfest order
3. Excited-state quantum phase transitions

\[ H = \frac{1}{2} n_d - \frac{1}{2N} \left( Q_{\chi=-\sqrt{7}/2} \cdot Q_{\chi=-\sqrt{7}/2} \right) \]

Finite-\(N\) realization:

Types of phase transitions:

A... saddle point  
continuous no order

B... local maximum  
2nd order

C... asymptotic value  
???
3. Excited-state quantum phase transitions

\[ H = \eta n_d - \frac{1}{N} (1 - \eta) (Q_{\chi = -\frac{\sqrt{7}}{2}} \cdot Q_{\chi = -\frac{\sqrt{7}}{2}}) \]

\[ \chi \]

More complex structures of the potential energy surface around the 1\textsuperscript{st} order QPT. Their influence on the spectrum was not yet studied in the IBM, but only in simpler models.
3. Excited-state quantum phase transitions

The 1D cusp Hamiltonian is given by:

\[ H = -\frac{\hbar^2}{2} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

where \( a = -1 \) and \( \hbar = 10^{-2} \).

The Hamiltonian has local maxima and minima, with the potential energy curves for different values of \( b \) shown in the upper right diagram.

- \( b = -\frac{1}{4} \) is a case of a 1st order QPT around a ground-state transition.
- \( b = 0 \) has a continuous no order transition.
- \( b = +\frac{1}{4} \) is a 2nd order transition.

The figure illustrates the potential energy curves and wave functions for different values of \( b \). The wave functions are labeled with a probability density squared, \( |\psi_i|^2 \).

Key points:
- **1st order QPT**
- **Local maximum**
- **Secondary minimum**

The reference is Cejnar, Stránský, PRE 78, 031130 (2008).
3. Excited-state quantum phase transitions

**1D cusp Hamiltonian**

\[ H = -\frac{\hbar^2}{2} \frac{d^2}{dx^2} + x^4 + ax^2 + bx \]

around 2nd order ground-state QPT

Cejn, Stránský, PRE 78,031130 (2008)

\[ b = 0 \]
\[ \hbar = 10^{-2} \]

![Graphical representation of the Hamiltonian with phase transitions and critical points]
3. Excited-state quantum phase transitions

$$H = (1 - \xi) n_b - \frac{\xi}{N} (Q \cdot Q)$$

$$Q = s^+ b_\lambda + b_\lambda^+ s$$

2D vibron model

- Around 2\textsuperscript{nd} order ground-state QPT
- Pérez-Bernal, Iachello (2008)
- Caprio, Cejnar, Iachello (2008)

Lipkin model (1D)

- Heiss, Müller (2002)
- Leyvraz, Heiss (2005)

Vibron model (3D→2D→1D)

IBM O(6)-U(5) (5D→2D→1D)

- Cejnar, Heinze, Jolie, Macek, Dobeš (2006, 2007)

Fermion pairing model (1D)

- Reis, Terra Cunha, Oliviera, Nemes (2005)
- Caprio, Cejnar, Iachello (2008)
Conclusions

The study of shape phase transitions in nuclei and related models can enrich the general theory of quantum phase transitions.

1. The influence of additional degrees of freedom
   Creation of new phases and phase transitions in combined systems. \( \otimes \) or \( \oplus \) cases

2. Mechanisms of first-order & continuous QPTs
   Asymptotic behavior of complex branch points near the real axis of the model control parameter determines the type of the phase transition.

\[ \text{1st order} \Leftrightarrow \text{locally 2-level mechanism} \]
\[ \text{continuous} \Leftrightarrow \text{many-level mechanism} \]

3. Quantum phase transitions for excited states
   Non-analytic changes observed in spectra of excited states. These changes are rooted in classical dynamics of the system.

\[ \text{singular growth of level density} \Leftrightarrow \text{singular evolution of the spectrum with control parameter} \]